ON THE SPECTRUM OF ALGEBRAIC $K$-THEORY

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ABSTRACT. The groups $K_i(A)$ of Bass for $i < 0$ are identified as homotopy groups of the spectrum of algebraic $K$-theory. The spectrum itself is identified. Applications to Laurent polynomials and to $K$-theory exact sequences are given.

Quillen has recently proposed a $K$-theory for unital rings [12], [13]. He associates to a ring $A$ a space $BG_1(A)^+$ whose homology is that of the group $G_1(A)$ and whose homotopy groups $\pi_i BG_1(A)^+$ he defines as $K_i(A)$, $i \geq 1$. The space $BG_1(A)^+$ is known to be an $H$-space, and indeed an infinite loop space.

Hence one is motivated to define $K_i(A)$, for $i \in \mathbb{Z}$, as $\pi_i(E(A))$, where $E(A)$ is the associated $\Omega$-spectrum. This note describes $E(A)$ and identifies the groups $K_i(A)$, $i < 0$. In fact, we show that the groups $K_i(A)$ are exactly the groups $L^{-i}K_0(A)$ discussed in Bass' book [3, p. 664] for $i < 0$.

Recall from the work of Karoubi and Villamayor [10] the cone $CA$ and suspension $SA$ of a ring $A$. An infinite matrix is called permutant if it is an infinite permutation matrix times a diagonal matrix of finite type. The diagonal matrix is of finite type if its diagonal entries are chosen from a finite subset of the ring. The ring $CA$ is the ring generated by permutant matrices. The cone $CA$ contains the two-sided ideal $\bar{A} = \bigcup_n M_n(A)$ and the quotient ring is called the suspension of $A$. We can now state our main result.

THEOREM A. The space $\Omega (BG_1(SA)^+)$ has the homotopy type of $K_0(A) \times BG_1(A)^+$.

COROLLARY. For all $i \in \mathbb{Z}$ we have $K_i(A) = K_{i+1}(SA)$.

Since Karoubi [9] has already identified $K_0(S'\bar{A})$ with Bass' groups $K_{-1}(A)$, the Corollary above completes the identification of Bass' groups with the negative homotopy of the spectrum $E(A)$.

In proving Theorem A we must first analyze the cone construction.

THEOREM B. The space $BG_1(CA)^+$ is contractible.

This result generalizes work of Karoubi and Villamayor [11] who show that $K_i(CA) = 0$ for $i \leq 2$. To prove Theorem B we observe that it suffices


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to show that $\check{H}_*(BG_1(CA)^+)$ = 0. This latter result is a consequence of a theorem of Karoubi [7] that the category $P(CA)$ is "flasque," together with a recent result of Barratt and Priddy [2] which implies

**Theorem C.** Suppose that $M$ is a locally free simplicial monoid (i.e. $M_n$ is a free monoid for all $n$). Let $\hat{M}$ denote the group completion of $M$. If $H_*(M, Z)$ is (graded) commutative, then $\pi_0(\hat{M}) = (\pi_0(\hat{M}))$ and $H_*(\hat{M}, Z) = H_*(M, X) \otimes_{Z[n_0(M)]} Z[\pi_0(\hat{M})]$.

D. W. Anderson informed me that Quillen has also given a proof of Theorem C.

In order to apply Theorem C, it is necessary to use the description of $BG_1(CA)^+$ given by D. W. Anderson [1] as the group completion of the morphism complex of the "blown-up" permutative category $P(CA)$.

Let $I$ be the image of $G_1(CA)$ in $G_1(SA)$. Since $K_1(CA) = 0$ it follows that $I = E(SA)$, the elementary group of matrices. We have two short exact sequences of groups

\[ G_1(\tilde{A}) \rightarrow G_1(CA) \rightarrow E(SA), \]

and

\[ E(SA) \rightarrow G_1(SA) \rightarrow K_1(SA). \]

Recall $Z_\infty$, the integral completion functor of Bousfield and Kan [4], [5]. We remark that $Z_\infty BG_1(A) \simeq BG_1(A)^+$, as was established in [6]. The following theorem was communicated to us directly by Bousfield.

**Theorem D (A. K. Bousfield).** Let $F \rightarrow E \rightarrow B$ be a fibration of connected spaces and suppose (1) $Z_\infty E$ and $Z_\infty B$ are nilpotent and (2) $\pi_1(B)$ acts nilpotently on each $H_n(F)$. Then the inclusion of $Z_\infty F$ in the fibre of the map $Z_\infty \pi$ is a homotopy equivalence, and moreover $Z_\infty F$ is nilpotent.

One checks the hypotheses of Theorem D for the fibration

\[ BE(SA) \rightarrow BG_1(SA) \rightarrow BK_1(SA). \]

The action of $K_1(SA)$ on $H_*(BE(SA))$ is the limit of inner automorphisms, and hence is trivial. We deduce that we have a fibration

\[ Z_\infty BE(SA) \rightarrow Z_\infty BG_1(SA) \rightarrow BK_1(SA) \]

with $Z_\infty BE(SA)$ nilpotent. In fact, $Z_\infty BE(SA)$ is easily seen to be the universal cover of $BG_1(SA)^+$. 

Consider now the commutative diagram whose rows are fibrations

\[
\begin{array}{ccc}
F_0 & \rightarrow & Z_\infty BG_1(CA) \\
\downarrow & & \downarrow \\
F & \rightarrow & Z_\infty BG_1(SA).
\end{array}
\]
After examining the exact homotopy sequences, it follows that \( F_0 \) is a connected component of \( F \). One considers now the fibration

\[
BG_1(\tilde{A}) \to BG_1(CA) \to BE(SA).
\]

Again one checks that the hypotheses of Theorem D are satisfied. Here the action of \( E(SA) \) on \( H_*(BG_1(\tilde{A})) \) is trivial. The idea for this observation is already contained in [9] in the proof of Lemma 5.9. It follows that \( Z_\infty BG_1(\tilde{A}) \simeq F_0 \). Next we establish

**Lemma.** \( Z_\infty BG_1(\tilde{A}) \simeq Z_\infty BG_1(A) \).

This Lemma generalizes [11, Proposition 7.4]. After an application of Theorem B, the proof of Theorem A is quickly completed.

Following Karoubi [9] we observe that the Corollary of Theorem A has application to the study of Laurent polynomials.

**Theorem E.** For any unital ring \( A \) and \( n \in \mathbb{Z} \) we have

\[
K_n(A[t, t^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus ?.
\]

One considers the pairing

\[
K_{n-1}(A) \otimes K_1(Z[t, t^{-1}]) \to K_n(A[t, t^{-1}])
\]

which induces a map \( K_{n-1}(A) \to K_n(A[t, t^{-1}]) \) by \( x \mapsto x \cup [t] \). Karoubi’s idea to exhibit an inverse to this map is as follows. He defines a homomorphism \( A[t, t^{-1}] \to SA \) by sending \( \sum a_it^i \) to the coset of the matrix in \( CA \):

\[
\begin{bmatrix}
  a_0 & a_{-1} & a_{-2} & \ldots \\
  a_1 & a_0 & a_{-1} & \ldots \\
  a_2 & a_1 & a_0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

This induces the map \( K_n(A[t, t^{-1}]) \to K_n(SA) \). By the Corollary to Theorem A, \( K_n(SA) = K_{n-1}(A) \) and one checks that the resulting map is a left inverse to the cup product map.

We also have results giving a homotopy theoretic interpretation to the \( K \)-theoretic exact sequences of a surjection. These \( K \)-theory sequences may be deduced from

**Theorem F.** Let \( q \) be a two-sided ideal in the ring \( A \). Let \( F \) be the homotopy theoretic fibre of the induced map \( BG_1(A)^+ \to BG_1(A/q)^+ \). Then there is a canonical isomorphism \( \pi_1 F \cong K_1(A, q) \).
I now know $K$-theoretic interpretations for the higher homotopy groups of the fibre $F$. It would be very interesting to interpret the corresponding fibre in the case of a localization $A \to A_S$.

BIBLIOGRAPHY


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