COHERENCE FOR CATEGORIES WITH ASSOCIATIVITY, COMMUTATIVITY AND DISTRIBUTIVITY

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Let \( \mathbb{C} \) be a category, \( \otimes, \oplus : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) two functors, \( U \) and \( N \) objects of \( \mathbb{C} \). Suppose that for any objects, \( A, B, C \) of \( \mathbb{C} \) we have natural isomorphisms,

\[
\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,
\]

\[
\alpha'_{A,B,C} : A \oplus (B \oplus C) \to (A \oplus B) \oplus C,
\]

\[
\lambda_A : U \otimes A \to A,
\]

\[
\lambda'_A : N \oplus A \to A,
\]

\[
\rho_A : A \otimes U \to A,
\]

\[
\rho'_A : A \oplus N \to A,
\]

\[
\gamma_{A,B} : A \otimes B \to B \otimes A,
\]

\[
\gamma'_{A,B} : A \oplus B \to B \oplus A,
\]

\[
\lambda^*_{A,B,C} : A \otimes N \to N,
\]

\[
\rho^*_{A,N} : A \otimes N \to N,
\]

(1)

and natural monomorphisms

\[
\delta_{A,B,C} : A \otimes (B \oplus C) \to A \otimes B \oplus A \otimes C,
\]

\[
\delta'_{A,B,C} : (A \oplus B) \otimes C \to A \otimes C \oplus B \otimes C.
\]

(II)

Roughly speaking the coherence problem is to determine the conditions (denoted coherence conditions) in which the arrows obtained by combining elements of type (I), (II) and identities with \( \otimes \) and \( \oplus \) only depend on the domain and codomain of the arrow. This note is to announce an answer to this question that was proposed in [6] as raised by H. Bass.

The first coherence results stated as such are contained in [5] which treats the case of only one functor \( \otimes \). The solution for a more complicated situation in closed categories is given in [4]. Other papers with results related to coherence problems are listed in the references.

* * *

Take a set, \( X = \{x_1, x_2, \ldots, x_p, n, u\} \), and construct \( \mathcal{A} \), the free \( \{\cdot, +\}\)-algebra over it. Let \( \mathcal{G} \) be the graph of all formal symbols, for \( x, y, z \in \mathcal{A} \),

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Construct \( H \), the free \( \{\cdot, +\} \)-algebra over \( G \) and take on \( H \) the only extension of the graph structure of \( G \) in which the projections are \( \{\cdot, +\} \)-morphisms. One element of \( H \) is said to be an instantiation if, with at most one exception, only elements of \( G \) of type \( 1_x \) occur in its expression. We will denote by \( J \) the graph of all the instantiations of \( G \).

Fix now \( p \) objects, \( C_1, C_2, \ldots, C_p \), of \( G \) and let \( g: J \to G \) be the morphism of graphs defined on the vertices by the conditions (i) \( gu = U \), \( gn = N \), \( gx_i = C_i \), for \( 1 \leq i \leq p \), (ii) \( g(x + y) = gx + gy \), \( g(xy) = gx \otimes gy \), for \( x, y \in J \), on \( G \) by taking each formal symbol onto the arrow of \( C \) determined replacing each subscript by its image by \( g \) and such that for \( x, y \in J \), \( g(x + y) = gx + gy \), \( g(xy) = gx \otimes gy \). This definition depends upon the \( C_i \) and allows us to define the value of a path with steps in \( J \) as the product of the images of the steps.

Let \( A \) be the free \( \{\cdot, +\} \)-algebra over \( X \), with associativity and commutativity for \( \cdot \) and \( + \), distributivity of \( \cdot \) relatively to \( + \), null element \( n \), identity element \( u \), and the additional condition \( na = n \) for \( a \in A \). The identity map of \( X \) defines a \( \{\cdot, +\} \)-morphism \( f: A \to A \). An element, \( a \), of \( A \) is defined to be regular if \( fa \) can be expressed as a sum of different elements of \( A \), each of which is a product of different elements of \( X \).

Our coherence conditions require the commutativity of the diagrams of a finite family of types. Roughly speaking our conditions are equivalent to the commutativity of any diagram that can be constructed taking, for each vertex, the iteration by \( \otimes \) and \( \oplus \) of not more than four objects, equal or different, of \( G \) and such that each edge is an iteration by \( \otimes \) and \( \oplus \) of arrows of type (I), (II) and identities: we are reduced to a finite number of types of diagrams if we drop unnecessary commutativity conditions.
With the above definitions we can state the following theorem which is our main result.

Coherence Theorem. If \( \mathcal{C} \) satisfies the coherence conditions and \( a \) is regular then the value of any path from \( a \) to \( b \), whose steps are in \( \mathcal{I} \), depends only upon \( a \) and \( b \).

A detailed exposition of these results will appear elsewhere.

References

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