LIMIT THEOREMS FOR CONTINUOUS STATE BRANCHING PROCESSES WITH IMMIGRATION

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Introduction. K. Kawazu and S. Watanabe [5] have defined a CBI process as a Markov process \( X = (x_t, P_x) \) with state space \([0, \infty)\), with \( \infty \) as a trap, possessing the property that, for each \( t \geq 0, \lambda \geq 0 \), there exist \( \phi(t, \lambda) \geq 0 \) and \( \psi(t, \lambda) \geq 0 \) such that

\[
E_x[e^{-\lambda x_t}; t < e_\infty] = \phi(t, \lambda)e^{-\psi(t, \lambda)},
\]

for every \( x \in [0, \infty] \); here \( e_\infty = \inf\{t: x_t = \infty\} \). Previously Lamperti [6] had treated the case \( \lambda > 1 \).

The Markov property of \( X \) implies that, for \( \lambda \geq 0, s, t \geq 0 \),

\[
\psi(t + s, \lambda) = \psi(t, \psi(s, \lambda)),
\]

\[
\phi(t + s, \lambda) = \phi(t, \lambda)\phi(s, \psi(t, \lambda)).
\]

Under the condition of right continuity of \( X \) at \( t = 0 \), it follows from (1.2) and (1.3) that \( \psi \) and \( \phi \) are differentiable. Explicitly, we have

\[
\frac{\partial \psi}{\partial t} = R(\psi), \quad \psi(0^+, \lambda) = \lambda,
\]

\[
\phi(t, \lambda) = \exp\left(-\int_0^t F(\psi(s, \lambda)) \, ds\right)
\]

for appropriate functions \( R \) and \( F \). Kawazu and Watanabe have used the property (1.1) to show that they must have the form

\[
R(\lambda) = -\alpha \lambda^2 + \beta \lambda + \gamma - \int_{0^+}^{\infty} \left(e^{-\lambda x} - 1 + \frac{\lambda x}{1 + x^2}\right)n_1(dx),
\]

\[
F(\lambda) = c + d\lambda - \int_{0^+}^{\infty} (e^{-\lambda x} - 1)n_2(dx),
\]

when \( n_1 \) and \( n_2 \) are measures on the Borel sets of \((0, \infty)\) with the property that

\[
\int_0^{\infty} \frac{u^2}{1 + u^2} n_1(du) + \int_0^{\infty} \frac{u}{1 + u} n_2(du) < \infty; \quad \alpha \geq 0, \gamma \geq 0, c \geq 0, d \geq 0.
\]

Furthermore any set of parameters \((\alpha, \beta, \gamma, c, d, n_1, n_2)\) define a unique
CBI process. In this note we shall only deal with conservative processes. According to Kawazu and Watanabe, this is equivalent to \( \gamma = c = 0 \) and \( \int_0^\infty R^*(\lambda)^{-1} \, d\lambda = +\infty \) where \( R^*(\lambda) = \max(R(\lambda), 0) \). This is satisfied, for instance, in the case \( \int_0^\infty x n_1(dx) < \infty \), which we shall explicitly assume; let

\[
\rho = R'(0) = \beta - \int_0^\infty \frac{x^3}{1 + x^2} n_1(dx).
\]

We shall first give a general result and then proceed to examine special cases.

**Statement of results.**

**Theorem.** Let \( X = (x_t, P_x) \) be a conservative CBI process with \( \int_1^\infty x n_1(dx) < \infty \). Let \( \rho(t) = e^{\rho t} \) if \( \rho > 0 \); \( \rho(t) = 1 \) if \( \rho \leq 0 \). As \( t \to \infty \), \( x_t/\rho(t) \) converges in distribution to a proper random variable if and only if

\[
(A) \quad \int_0^1 \frac{F(\lambda)}{|R(\lambda)|} \, d\lambda < \infty.
\]

**Corollary 1.** Let \( \rho > 0 \), \( \int_1^\infty x \log x n_1(dx) < \infty \). Then as \( t \to \infty \), \( x_t/\rho^t \) has a proper, nondegenerate limiting distribution if and only if

\[
(B) \quad \int_1^\infty (\log x)n_2(dx) < \infty.
\]

The convergence takes place almost surely and in \( L^1 \) mean.

**Corollary 2.** Let \( \rho < 0 \). Then as \( t \to \infty \), \( x_t \) has a proper, nondegenerate limiting distribution if and only if \( (B) \) is satisfied.

**Corollary 3.** Let \( \rho = 0 \), \( \int_1^\infty x^2 n_1(dx) = \infty \). Then as \( t \to \infty \), \( x_t \) has a proper, nondegenerate limiting distribution if and only if \( (A) \) is satisfied.

For comparison with known theorems for Galton-Watson processes we give the following result which applies in the case of finite variance.

**Corollary 4.** Let \( \rho = 0 \), \( \int_1^\infty x^2 n_1(dx) < \infty \), \( \int_1^\infty x n_2(dx) < \infty \); then as \( t \to \infty \), \( x_t/t \) has a proper, nondegenerate limiting distribution.

A short calculation shows that the condition \( (B) \) is a special case of the condition \( (A) \) for the case \( R(\lambda) = \lambda \). This case appeared \([1]\) in the study of a storage system proposed by Moran \([7]\). Condition \( (B) \) has appeared in the study of discrete parameter, discrete state branching processes, by Heathcote \([3]\), \([4]\). Corollary 3 is related to a result of Seneta \([8]\). Recent work of Foster and Williamson \([2]\) extend Seneta’s observations.

When condition \( (B) \) fails, the following result gives a nonlinear normalization which produces weak convergence. We know of no analogue in the
discrete parameter case. For simplicity, we state the result in the subcritical case.

**Theorem 2.** Let $X = (x_t, P_x)$ be a conservative CBI process with $-\infty < \rho < 0$. For $x > 0$ let

$$H(x) = \int_{e^{-x}}^{1} F(u) \frac{du}{R(u)}, \quad m(x) = \exp(H(\log x)).$$

Assume that as $x \to \infty$, we have

(C1) $H(x) \to \infty$,  
(C2) $xH'(x) \to 0$.

Then for $0 \leq u \leq 1$,

(*) $P_x\{m(x_t)/m(e^{ct}) \leq u\} \to u^{1/c}$,

as $t \to \infty$, here $c = -\rho < 0$.

This result covers cases in which the integral (B) diverges "slowly." Condition (C2) holds, for example, if $H(x) = \log \log x$; then $(\log \log x_t)/(\log ct)$ converges weakly to a limit. If $H(x) = \log x$, condition (C2) fails; a direct calculation shows nonetheless that we have $(\log x_t)/ct$ weakly convergent when $t \to \infty$. If $H(x) = x^{1/2}$, a direct calculation shows that, as $t \to \infty$, $(\log x_t)/t^2$ converges weakly; this is not of the above form (*).

Professor Michael B. Marcus has made the useful observation that $H(x)$ can be expressed directly in terms of the distribution $n_2$ by the relation

$$H(x) \sim \text{const} \int_{[u, \infty)} \frac{f_2}{u} \, du.$$  

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**References**


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