INVARIANT SPLITTINGS IN NONASSOCIATIVE ALGEBRAS: A HOPF APPROACH

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The purpose of this short note is to announce generalizations of known invariant splitting theorems due to Taft [4], [5], [6], [7] and Mostow [1], which have been obtained by Hopf methods. The approach is an outgrowth of techniques developed by M. Sweedler in order to study algebraic groups from a Hopf point of view, and was motivated by several conversations with him.

0. Let \((V, \Delta, \varepsilon)\) be a coalgebra over the field \(k\) which is equipped with the structure of a unitary associative algebra by means of coalgebra morphisms \(m: V \otimes_k V \to V\) and \(\mu: k \to V. A = (V, \Delta, \varepsilon, m, \mu)\) is then a bialgebra and is a Hopf algebra if \(\text{id} \in \text{End}_k(V)\) is invertible in the convolution structure [2, p. 71]. We will often confuse \(A\) with \(V\).

Recall that \(A^*\) has a natural associative algebra structure relative to \(A^* \otimes_k A^* \to (A \otimes_k A)^* \Delta_* A^*, \quad k \cong k^* \otimes A^*\).

An element \(\lambda \in A^*\) is called a (left) integral for \(A\) if \(a^* \lambda = \langle a^*, 1_A \rangle \lambda\) for all \(a^* \in A^*\). If \(M \otimes_k A\) is a right \(A\)-comodule, then \(M\) carries a (rational) left \(A^*\)-module structure via

\[
A^* \otimes_k M \to A^* \otimes_k M \otimes_k A \to M \otimes_k A^* \otimes_k A \to M \otimes_k k \cong M
\]

[2, pp. 33–36, 91–92] and one has the adjoint \(A^*\)-module structure on \(E = \text{End}_k M\) given in [3, p. 332] which is characterized by the relation

\[
(a^* - T)(m) = \sum_{(m)} (a^* - m_{(1)}) \cdot T(m_{(0)}) \quad \text{for } a^* \in A^*, \; T \in E \; \text{and } m \in M.
\]

If \(A\) has an integral \(\lambda\) which satisfies \(\langle \lambda, 1_A \rangle = 1\), then every rational \(A^*\)-module is completely reducible. Conversely, if \(\lambda A\) is a completely reducible rational \(A^*\)-module (via the regular right \(A\)-comodule structure of \(A\)) then \(A\) has an integral satisfying the above condition.

1. Let \(\mathfrak{N}\) be a nonassociative algebra over \(k\), \(\mathfrak{R}\) an ideal in \(\mathfrak{N}\) with \(\mathfrak{N}\mathfrak{R} = \{0\}, \mathfrak{E}\) a subalgebra of \(\mathfrak{N}\) with \(\mathfrak{N} = \mathfrak{E} \oplus \mathfrak{R}\) (as vector spaces). We have

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THEOREM. Let $A$ be a commutative Hopf algebra and $\psi : \mathcal{R} \to \mathcal{R} \otimes_k A$ a comodule structure map which is multiplicative. Assume further that $A^*A$ is completely reducible and that $\mathcal{R}$ is a subcomodule. Then there is a subalgebra of $\mathcal{R}$ which is a subcomodule and a vector space complement to $\mathcal{R}$.

2. Throughout this section $\mathcal{R}$ is a nonassociative algebra and a right comodule for a commutative Hopf algebra $A$ where the comodule structure map $\psi$ is multiplicative and $A^*A$ is completely reducible. Using the preceding result one easily obtains the following:

THEOREM EA. If $\mathcal{R}$ is a finite-dimensional associative algebra which is separable modulo its radical $\mathcal{R}$, and $\mathcal{R}$ is an $A$-subcomodule, then there is a subalgebra of $\mathcal{R}$ which is a subcomodule and vector space complement to $\mathcal{R}$.

THEOREM EL. If $\mathcal{R}$ is a finite-dimensional Lie algebra over a field of characteristic 0, and $\mathcal{R} = \text{radical } \mathcal{R}$ is a subcomodule, then there is a subalgebra of $\mathcal{R}$ which is a subcomodule and vector space complement to $\mathcal{R}$.

One has similar results for the case of alternative or Jordan algebras.

3. In the notation of §2 we let $\mathcal{S}$ be a subalgebra subcomodule complement to $\mathcal{R}$ and $\mathcal{S}_1$ any separable subalgebra subcomodule of $\mathcal{R}$. For $\mathcal{B} \subseteq \mathcal{R}$ we let $B^* = \{v \in \mathcal{B} | a^* \cdot v = \langle a^*, 1 \rangle v, \text{ for all } a^* \in A^*\}$. We have

THEOREM UA. Under the hypothesis of EA there is an $x \in \mathcal{R}^*$ such that conjugation by $1 + x$ induces a comodule morphism carrying $\mathcal{S}_1$ into $\mathcal{S}$.

THEOREM UL. Under the hypothesis of EL, there is an $x \in (\text{Nil } \mathcal{R})^*$ (Nil $\mathcal{R}$, the nilradical of $\mathcal{R}$) such that $\exp(adx)$ induces a comodule morphism carrying $\mathcal{S}_1$ into $\mathcal{S}$.

One has results similar to those in [7] for the case of alternative or Jordan algebras.

BIBLIOGRAPHY


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