SURFACES IN CONSTANT CURVATURE MANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD

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1. Statement of results. For an (n)-dimensional Riemannian manifold $M^n$, isometrically immersed in an $(n + k)$-dimensional Riemannian manifold $M^{(n+k)}$ of constant sectional curvature $c$, let $H$ denote the mean curvature vector field of $M^n$. $H$ is a section of the normal bundle $NM^n$ of the immersion. When $n = 2$, $k = 1$, and $c = 0$ (a surface immersed in $E^3$), the requirement $|H| = \text{constant}$ is classical constant mean curvature. If $k > 1$, however, the condition $|H| = \text{constant}$ is usually too weak to prove reasonable generalizations of the classical theorems for surfaces of constant mean curvature in $E^3$. We investigate a stronger requirement on $H$; namely, that $H$ be parallel with respect to the induced connection in the normal bundle (for definitions, see II). Then using an analytic construction first employed by H. Hopf [2], we obtain

**Theorem 1.** A compact surface $M^2$ of genus 0 immersed in $M^4(c), c \geq 0$, upon which $H$ is parallel in the normal bundle, is a sphere of radius $1/|H|$.

This generalizes a theorem of Hopf, who proved that the only immersed surfaces in $E^3$ of genus 0 with constant mean curvature are spheres [2, Chapter 7, §4]. For complete surfaces in $E^4$, we prove

**Theorem 2.** A complete surface $M^2$, immersed in $E^4$, whose Gauss curvature does not change sign, and for which $H$ is parallel in the normal bundle $NM^2$, is a minimal surface ($H \equiv 0$), a sphere, a right circular cylinder, or a product of circles $S^1(r) \times S^1(\rho)$, where $|H| = \frac{1}{2}(1/r^2 + 1/\rho^2)^{1/2}$.

This extends a theorem of Klotz and Osserman for complete surfaces of constant scalar mean curvature in $E^3$ [5]. It can also be generalized to immersions into $M^4(c), c \geq 0$. Theorem 2 is proved in two steps. First we prove

**Theorem 3.** A piece of immersed surface $M^2$ in $E^4$, satisfying the conditions of Theorem 2 with $H \neq 0$, is either a sphere or it is flat ($K = 0$).

Then we establish the following characterization of flat surfaces in $E^4$ with parallel mean curvature vector fields:


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Theorem 4. A piece of immersed surface $M^2$ in $E^4$ with parallel mean curvature vector $H \neq 0$ and $K \equiv 0$ is a piece of $S^1(r) \times S^1(p)$: the product of two circles of radius $r$ and $p$ with the standard flat immersion. ($p$ may be infinite, in which case we have a right circular cylinder.)

Theorem 2 generalizes to immersions in $S^4(c)$.

Surfaces in $E^n$ which lie minimally in hyperspheres of radius $r$ have the same mean curvature vectors as the hypersphere, and consequently have parallel mean curvature vector fields. Such surfaces are pseudo-umbilic ($\varphi_3 \equiv 0$ in the lemma in II). In this case, Itôh [3] has proven a special case of Theorem 2 for immersions in $E^4$ (see also Chen, [1]). For minimal surfaces in $S^4$, Ruh [8] has proven a case of Theorem 1, using methods similar to the basic lemma in II. For a wide variety of examples of complete minimal surfaces in $S^3$, see Lawson [6].

It is possible to show the existence of surfaces in $E^n$ and $S^n(c)$ with parallel $H$ and $(p \neq 0)$ (i.e. they do not lie minimally in hyperspheres). The method employed uses a theorem due to Szczarba [9] on existence of immersions in constant curvature manifolds with codimension $k > 1$.

II. Definitions and Main Lemma. $\nabla$ denotes covariant differentiation on $M^{n+k}$, and $\nabla$ denotes covariant differentiation on $M^n \subset M^{n+k}$. For $X, Y$, sections of $TM^n$, $\nabla_X Y = [\nabla_X Y]^T$ where $[\ ]^T$ is projection onto $TM^n$. $[\ ]^N$ is projection onto $NM^n$.


$D$ defines a connection on $NM^n$. $A(X, N) = [\nabla_X N]^T$. $A$ is a tensor: $A_p: TM^n \times NM^n \rightarrow TM^n$ is bilinear.

For an orthonormal framing $(e_1, \ldots, e_n)$ of $TM^n$, $H = (1/n)\sum_{i=1}^n B(e_i, e_i)$. This definition of $H$ is independent of the framing. A normal vector field $N$ is said to be parallel in the normal bundle $NM^n$ if $D_X N = 0$ for all $X$ in $TM^n$. This condition implies $|N| = \text{const.}$ Thus our assumption that $H$ is parallel in $NM^n$ includes constant mean curvature. ($|H| = c$.)

The Gauss and Codazzi equations, for $X, Y, Z$ sections in $TM^n$, are

1. $R(X, Y)Z = c\{\langle X, Z \rangle Y - \langle Y, Z \rangle X \} + A(X, B(Y, Z)) - A(Y, B(X, Z))$,
2. $(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z),$

where $(\nabla_X B)(Y, Z) = D_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$ (for a reference for the above definitions and equations, see [4, Chapter 7]).

For $X, Y$ in $TM^n$, $N$ in $NM^n$, $\bar{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{\langle XY \rangle} N$ is the curvature tensor for $D$. For $\bar{R}$, there is a Gauss-type equation

3. $\bar{R}(X, Y)N = B(X, A(N, Y)) - B(Y, A(N, X))$

and an equation, analogous to (2),

For a unit normal vector \( e_a \) at \( p \in M^n \), the matrix \( \left( \lambda_{ij}^3 \right) = (B(e_i, e_j) \cdot e_a) \) is the "second fundamental form matrix in the \( x \) direction." We specify \( H/|H| \) as \( e_{n+1} \) when \( H \neq 0 \). Considering an immersed surface \( (n = 2) \) given in conformal coordinates \((u, v)\): \( ds^2 = E(du^2 + dv^2) \), we associate to it a natural framing

\[
(e_1, e_2) = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \sqrt{E} \right)
\]

of the tangent bundle, \( TM^2 \).

**Lemma.** For an immersed surface, \( M^2 \subset \mathbb{R}^{n+k} \) in conformal coordinates, let \( H \neq 0 \) and \( e_a \) be a unit normal vector field with \( e_a \cdot H = 0 \):

(a) if \( H \) is parallel in \( NM^2 \), then \( \varphi_3 = E\{ \frac{1}{2}(\lambda_1^3 - \lambda_2^3) - i\lambda_1^3 \} \) is an analytic function of \( z = u + iv \);

(b) if \( e_a \) is parallel in \( NM^2 \), then \( \varphi_a = E\{ \lambda_1^3 - i\lambda_1^3 \} \) is an analytic function of \( z \); 

(c) if \( k = 2 \) and \( H \) is parallel, then \( e_a \) is parallel, and both \( \varphi_3 \) and \( \varphi_a \) are analytic;

(d) under the conditions of (a) and (b), either \( \varphi_3 \equiv 0 \) or \( \varphi_a = \kappa \varphi_3 \) where \( \kappa \) is a real constant.

**Sketch of Proof.** (a) Using equation (4) with \( X = \partial/\partial u \), \( Y = \partial/\partial v \), \( N = H \), and the assumption that \( H \) is parallel, the equations

\[
(E\lambda_1^3)_u - (E\lambda_2^3)_v = \frac{1}{2}E_u(\lambda_1^3 + \lambda_2^3), \quad (E\lambda_1^3)_v - (E\lambda_2^3)_u = \frac{1}{2}E_v(\lambda_1^3 + \lambda_2^3)
\]

are obtained. (5) is in the same form as the Codazzi equations in conformal coordinates for surfaces in \( E^3 \), only it is expressed for the distinguished normal \( e_3 = H/|H| \). Since \( \lambda_1^3 + \lambda_2^3 = 2|H| \) is constant, (5) reduces to the Cauchy-Riemann equations for \( \varphi_3 \).

(b) Proof follows that of (a), using the fact that \( \lambda_1^3 + \lambda_2^3 = 0 \).

(c) Since \( NM^2 \) is 2-dimensional, the assumption that \( H \) is parallel forces \( e_a \) to be parallel. Then (a) and (b) imply analyticity.

(d) Using equation (3) with

\[
X = \frac{\partial}{\partial u_1} \sqrt{E}, \quad Y = \frac{\partial}{\partial u_2} \sqrt{E}, \quad N = e_3,
\]

we obtain, using the fact that \( e_3 \) is parallel,

\[
0 = \left\{ \sum_{k=1}^2 \lambda_k^3 \lambda_k^3 - \lambda_k^3 \lambda_k^3 \right\}
\]

Note that (6) implies \( \text{Im}(\varphi_3) = 0 \). So if \( \varphi_3 \neq 0 \), \( \varphi_a/\varphi_3 = \varphi_a \cdot \varphi_3/|\varphi_3|^2 \) is real. By (a) and (b), it is also meromorphic, hence constant.

**III. Proof of Theorems (Sketch).** Theorem 1 is proved by constructing an analytic differential \( \theta_3 \) out of the functions \( \varphi_3(z) \) of the lemma: in local coordinates, \( \theta_3 = \varphi_3 dz^2 \). Since \( M^2 \) is of genus 0, \( \theta_3 \) must be identically zero.
Hence \( \varphi_3(z) \equiv 0 \). This implies that \( M^2 \) is pseudo-umbilic (\( \lambda^3_{11} = \lambda^3_{22}, \lambda^3_{12} = 0 \)). The function \( \varphi_4 \) associated with \( e_4, e_4 \cdot H = 0 \) is also zero by a similar argument. Hence \( M^2 \) is totally umbilic. This implies that \( M^2 \) is immersed as a sphere.

To prove Theorem 3, we can consider on \( M^2 \) the quadratic analytic differentials \( \varphi_3dz^2 \) and \( \varphi_4dz^2 \) (where \( \varphi_3, \varphi_4, \) and \( z \) are as in the lemma). If \( K \geq 0 \), \( M^2 \) is either compact or parabolic by a theorem of Huber (see [5, p. 316]). If it is compact, then either \( K \equiv 0 \) or \( M^2 \) is of genus 0. The genus 0 case is a sphere by Theorem 1.

If \( K \leq 0 \), then \( |H|^2 - K > |H|^2 > 0 \). In a neighborhood of each point, we introduce the new metric \( ds^2 = E(|H|^2 - K)^{1/2}(du^2 + dv^2) \). Using the equality

\[
|\varphi_3|^2 + |\varphi_4|^2 = E^2(|H|^2 - K)
\]

and part (d) of the lemma to show that \( \Delta \log(|\varphi_3|^2 + |\varphi_4|^2) = 0 \), we establish that \( ds^2 \) is a flat metric. Therefore, the universal covering surface \( \tilde{M}^2 \) of \( M^2 \) is conformally the plane. The function \( |H|^2 - K \) is easily seen to be superharmonic. Since it is bounded below, it must be constant. Hence \( K \) is constant, and must be zero.

Theorem 4 is proved by introducing conformal coordinates \((u, v)\) such that \( E \equiv 1 \). The lemma is used to show that all \( \lambda^3_{ij} \) are constant. Then a rotation of coordinates puts the immersion into the form

\[
(u, v) \rightarrow \left( r \cos \frac{u}{r}, r \sin \frac{u}{r}, \rho \cos \frac{v}{\rho}, \rho \sin \frac{v}{\rho} \right).
\]

The constants \( r \) and \( \rho \) are determined from the \( \lambda^3_{ij} \) and \( |H| \). This immersion is the standard flat immersion of the plane into \( E^4 \) as a product of circles.

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