INVARIANT SUBSPACE THEORY FOR THREE-DIMENSIONAL NILMANIFOLDS

BY LOUIS AUSLANDER¹ AND JONATHAN BREZIN²,³

Communicated by C. C. Moore, September 15, 1971

1. Introduction. Let $N$ denote the nilpotent Lie group whose underlying manifold is three-dimensional Euclidean space $\mathbb{R}^3$ and whose group operation is given by $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. The subset $\Gamma = \{(a, b, c) : a, b, c \in \mathbb{Z}\}$ of $N$ is a subgroup, and the quotient $N/\Gamma$ is a compact manifold, denoted $M$. On the manifold $M$ there is a unique probability measure $\nu$ invariant under translation by $N$. (We use right cosets $Tg$, $g \in N$, and hence translation here means right-translation.) We will use $R$ to denote the regular representation of $N$ on $L^2(M, \nu)$, namely: $(R_g \phi)(T h) = \phi(T gh)$ for all $g, h \in N$ and all $\phi \in L^2(M, \nu)$.

The representation $R$ decomposes into a direct-sum of irreducible subrepresentations. However, some of the irreducible representations in the sum occur with multiplicity greater than 1, and consequently, $L^2(M, \nu)$ does not decompose uniquely into a direct sum of irreducible $R$-invariant subspaces. The theorems announced below are aimed toward remedying this situation by introducing into the family of all irreducible $R$-invariant subspaces of $L^2(M, \nu)$ a certain amount of structure.

Let $3N$ denote the center of $N$. The Stone-von Neumann theorem says that for each nonzero real number $\xi$, there is a unique (up to unitary equivalence) irreducible unitary representation $U^\xi$ whose restriction to $3N$ is a multiple of the character $(0, 0, z) \mapsto e^{2\pi i z\xi}$ of $3N$. We will use $L(\xi)$ to denote the Hilbert space of $U^\xi$.

It is easy to see that, aside from the characters of $N$ vanishing on $\Gamma$, the only irreducible summands of $R$ are those $U^\xi$ where $\xi$ is a nonzero integer. In fact, let $n$ be a nonzero integer, and let $H(n)$ denote the subspace of $L^2(M, \nu)$ consisting of those functions $f$ satisfying $(R_{(0, 0, z)} f)(\Gamma h) = e^{2\pi i n z} f(\Gamma h)$ for all $h \in N$ and $(0, 0, z) \in 3N$; then the restriction of $R$ to $H(n)$ is unitarily equivalent to the representation $U^n \otimes 1$ of $N$ on $L(n) \otimes C^n$. (For a proof, see C. C. Moore [2].) It follows that the irreducible subspaces of $H(n)$ are in one-to-one correspondence with the space of lines in $C^n$ through 0—that is, projective space $CP^{n-1}$. The theorems below refine this observation.

AMS 1969 subject classifications. Primary 2265.

Key words and phrases. Nilmanifold, harmonic analysis.

¹ John Simon Guggenheim Fellow.
² Alfred P. Sloan Fellow.
³ Both authors partially supported by the National Science Foundation.

Copyright © American Mathematical Society 1972

255
2. Main results. In accord with the notation already established, we set \( H(0) \) equal to the subspace of \( L^2(M, \nu) \) consisting of those functions \( f \) constant on orbits of \( 3N \) in \( M \). Also, we set \( A = H(0) \cap C^\infty(M) \). We then have that \( A \) is a subalgebra of \( C^\infty(M) \), and that each \( H(n) \) becomes an \( A \)-module if we set \((af)(m) = a(m)f(m)\) for all \( a \in A, f \in H(n) \), and \( m \in M \).

**Theorem 1.** Let \( n \) be a nonzero integer, let \( K \) be an irreducible \( R \)-invariant subspace of \( H(n) \), and let \( A(K) = \{a \in A : a \cdot f \in K \text{ for all } f \in K\} \). Then \( A(K) \) is a subalgebra of \( A \) that is closed under complex conjugation, and \( A \) is a free \( A(K) \)-module whose dimension divides \( n^2 \) and is divisible by \( n \).

We define the index of \( K \), \( \text{ind}(K) \), to be the integer \( (\dim A(K))/|n| \).

Let \( V : H(n) \rightarrow L(n) \otimes C[n] \) be an isometric isomorphism that intertwines \( R \) and \( U^n \otimes 1 \). Let \( \xi \in CP^{[n]-1} \) and, thinking of \( \xi \) as a line in \( C[n] \), pick \( v \) from among the nonzero points on \( \xi \). Then \( V^{-1}(L(n) \otimes v) \) is an irreducible \( R \)-invariant subspace \( K(\xi) \) depending only on \( \xi \) and not on the choice of \( v \).

**Theorem 2.** Let \( n \) be a nonzero integer, and let \( d \) be a positive integer that divides \( n \). Then

\[
\{\xi \in CP^{[n]-1} : \text{ind}(K(\xi)) \leq d\}
\]

is a nonempty algebraic set of dimension \( \leq d - 1 \).

Theorem 2, in particular, says that \( \text{ind}(K(\xi)) = 1 \) for only finitely many \( \xi \in CP^{[n]-1} \). Our next result characterizes the \( K(\xi) \) with index 1. First, a definition:

Let \( C \) denote the subgroup \( \Gamma 3N \) of \( N \), and for each nonzero integer \( n \), let \( C_n \) denote the subgroup \( \{(a/n, b/n, z) : a, b \in \mathbb{Z}, z \in \mathbb{R}\} \) of \( N \). Let \( D \) be a subgroup of \( C_n \) that contains \( C \) as a subgroup of index \( n \), and let \( \chi_n \) denote the character \( (a, b, z) \rightarrow e^{2\piinz} \) of \( C \). It is not hard to see that \( \chi_n \) can be extended to a character \( \chi_n' \) of \( D \). The unitary representation of \( I_D \) of \( N \) induced by \( \chi_n' \) from \( D \) is irreducible by Mackey’s little-group theorem (see [1]). The representation \( I_D \) can be described as follows:

Let \( \mu \) denote Lebesgue measure on the torus \( N/D \), and let \( \eta : N/D \rightarrow N \) be a section. For all \( h \in N \), set \( X(h) = \chi_n'(\eta(h)^{-1}) \). Then \( I_D \) is given on \( L^2(N/D, \mu) \) by \( (I_D \phi)(Dh) = (X(hg)(X(h))\phi)(Dhg) \) for all \( h, g \in N \) and \( \phi \in L^2(N/D, \mu) \).

The function \( X \) is constant on right \( \Gamma \) cosets, and therefore we can map \( L^2(N/D, \mu) \) into \( H(n) \) by defining \( (W_D \phi)(\Gamma h) = X(h)\phi(Dh) \). With \( W_D \) so defined, we have \( W_D I_D = R_g W_D \) for all \( g \in N \). Hence the image in \( H(n) \) of \( W_D \) is an irreducible \( R \)-invariant subspace.
We shall say that an irreducible $R$-invariant subspace $K$ of $H(n)$ is \textit{rationally presentable} if for a suitable choice of $D$ and $\gamma_a$, the subspace $K$ is the image of the map $W^D$.

\textbf{Theorem 3.} Let $n$ be a nonzero integer, and let $K$ be an irreducible $R$-invariant subspace of $H(n)$. Then the following three conditions on $K$ are equivalent:

\begin{enumerate}
    \item $K$ is rationally presentable.
    \item $\text{ind}(K) = 1$.
    \item There is a function $f \in K$ such that $|f(m)| = 1$ for almost all $m \in M$ and such that $\{a \cdot f : a \in A(K)\}$ is dense in $K$.
\end{enumerate}

Making use of the subgroup $C_n$, we can introduce some structure into the family $Q(n)$ of rationally presentable subspaces of $H(n)$. We begin by observing that if $f \in H(n)$ and if $g \in C_n$, then the correspondence $\Gamma h \mapsto f(\Gamma g^{-1} h g)$ defines a new function, denoted $L_g f$, in $H(n)$. If $K$ and $K'$ are in $Q(n)$, and if $K = L_g K'$ for some $g \in C_n$, we shall call $K$ and $K'$ \textit{inner relatives}.

\textbf{Theorem 4.} Inner relatedness is an equivalence relation on $Q(n)$, and each equivalence class contains precisely $|n|$ elements. If $K_1 \in Q(n)$, and if $K_2, \ldots, K_{|n|}$ are the remaining inner relatives of $K_1$, then $K_1, \ldots, K_{|n|}$ are mutually orthogonal and $H(n) = \bigoplus_{j=1}^{|n|} K_j$.

For each nonzero integer $n$, define an epimorphism $\varepsilon_n : N \to N$ by $\varepsilon_n(x, y, z) = (x, ny, n z)$. Then $\varepsilon_n(\Gamma) \subseteq \Gamma$, and thus $\varepsilon_n$ induces $\varepsilon_n^* : M \to M$. Let $K_1^{(n)} = \{f \circ \varepsilon_n^* : f \in H(1)\}$. Then $K_1^{(n)} \in Q(n)$. Let $K_2^{(n)}, \ldots, K_{|n|}^{(n)}$ be the inner-relatives of $K_1^{(n)}$. Then $L_2(M, v) = H(0) \bigoplus \sum_{n \neq 0} \bigoplus_{j=1}^{|n|} K_j^{(n)}$. Using families of epimorphisms other than the family $\{\varepsilon_n\}$, we can generate other direct-sum decompositions of $L_2(M, v)$. One corollary of all of this is the following theorem:

\textbf{Theorem 5.} Let $f$ be a real-analytic function on $M$, and let $K$ be any irreducible $R$-invariant subspace of $L_2(M, v)$. Then the orthogonal projection of $f$ onto $K$ is also real-analytic.

Indeed, if $K$ is in $H(0)$, or is $H(1)$, Theorem 5 is obvious; the theorem follows in general by working with the spaces $K^{(n)}_j$.

We remark, in conclusion, that all of our results generalize without difficulty to 2-step nilpotent Lie groups in general. For more complicated nilpotent Lie groups, the situation at present is not very clear, and is being worked on.
REFERENCES


2. C. C. Moore, Decomposition of unitary representations defined by discrete subgroups of nilpotent groups, Ann. of Math. (2) 82 (1965), 146–182. MR 31 #5928.

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10036

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address: (L. Auslander) INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540