THE INNER PRODUCT OF PATH SPACE MEASURES 
CORRESPONDING TO RANDOM PROCESSES 
WITH INDEPENDENT INCREMENTS

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Let $X_1(t)$ and $X_2(t)$ be any two stochastically continuous, homogeneous random processes on $[0, T]$ with independent increments. It follows that $E(\exp(irX_k(t))) = \exp(tD_k(r))$, where

\begin{equation}
D_k(r) = ir_k - \frac{\beta_k}{2} + \int_{\mathbb{R}} \left( e^{iru} - 1 - \frac{iru}{1 + u^2} \right) d\sigma_k(u)
\end{equation}

for some $\alpha_k \in \mathbb{R}$, $\beta_k \geq 0$, and Borel measure $\sigma_k$ with $\int_{\mathbb{R}} u^2/(1 + u^2) \, d\sigma_k(u) < \infty$ (and with $\sigma_k(\{0\}) = 0$). We denote by $\hat{\rho}_k$ (resp. $\rho_k^s$) the probability measure on $\mathbb{R}$ with characteristic function, $\exp(D_k(r))$ (resp. $\exp(tD_k(r))$), and by $\hat{\rho}_k$ the probability measure on path space corresponding to $X_k$. $\hat{\rho}_k$ is a Borel measure (with respect to the Skorokhod topology) on $D = D[0, T]$, the space of real valued functions on $[0, T]$ which are right-continuous and have left-hand limits, and may be defined in terms of $\rho_k$ in the usual way.

If $\mu_1$ and $\mu_2$ are two measures on $\mathbb{R}$ (or $D$), we define $\sqrt{\mu_1 \mu_2}$ as the unique measure satisfying

\[ \frac{d(\sqrt{\mu_1 \mu_2})}{dv} = \sqrt{\frac{d\mu_1}{dv} \frac{d\mu_2}{dv}} \]

for any $v \gg \mu_1, \mu_2$; $(\sqrt{\mu_1} - \sqrt{\mu_2})^2$ thus denotes the (positive) measure, $(\mu_1 + \mu_2) - 2\sqrt{\mu_1 \mu_2}$. Given $\rho_1$ and $\rho_2$ as above, we define $N = N(\rho_1, \rho_2) = \int_{\mathbb{R}} d(\sqrt{\sigma_1} - \sqrt{\sigma_2})^2$; $N$ may be finite or infinite. If $N < \infty$, it is easily shown that $\int_{\mathbb{R}} u/(1 + u^2) \, d|\sigma_1 - \sigma_2| < \infty$ and we then define

\[ \gamma = \gamma(\rho_1, \rho_2) = \frac{1}{2} \left( \alpha_1 - \alpha_2 - \int_{\mathbb{R}} \frac{u}{1 + u^2} \, d(\sigma_1 - \sigma_2) \right). \]

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We define $K = K(\rho_1, \rho_2)$ as follows:

$$K = \frac{\gamma^2}{2\hat{\beta}} + \frac{N}{2}, \quad \text{if } N < \infty \text{ and } \beta_1 = \beta_2 = \beta > 0,$$

$$= \frac{N}{2}, \quad \text{if } N < \infty \text{ and } \beta_1 = \beta_2 = 0 \text{ and } \gamma = 0,$$

$$= + \infty, \quad \text{otherwise.}$$

When $K < \infty$, we define $\rho_3$ as the measure on $R$ with characteristic function $\exp(D_3(r))$, where $D_3$ is given by (1) with

$$\alpha_3 = \frac{\alpha_1 + \alpha_2}{2} - \frac{1}{2} \int_R \frac{u}{1 + u^2} d(\sqrt{\sigma_1} - \sqrt{\sigma_2})(u),$$

$\beta_3 = \beta$ (where $\beta = \beta_1 = \beta_2$), and $\sigma_3 = \sqrt{\sigma_1} \sqrt{\sigma_2}$. We let $\tilde{\rho}_3$ denote the related measure on $D$ corresponding to a third homogeneous random process with independent increments, $X_3$, in the obvious way.

**Theorem.** (i) If $K = \infty$, then $\sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2} = 0$.

(ii) If $K < \infty$, then $\sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2} = e^{-TK} \tilde{\rho}_3$.

Since $\tilde{\rho}_1 \perp \tilde{\rho}_2 \iff \sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2} = 0$, we immediately obtain

**Corollary 1.** $\tilde{\rho}_1 \perp \tilde{\rho}_2$ if and only if either

(i) $N = \infty$, or

(ii) $N < \infty$, but $\beta_1 \neq \beta_2$, or

(iii) $N < \infty$, and $\beta_1 = \beta_2 = 0$, but $\gamma \neq 0$.

When $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are not mutually singular, we wish to define a quantitative description of their "overlap." Accordingly, we consider for two measures, $\mu_1$ and $\mu_2$, on $D$ (resp. $R$) the decomposition of $D$ (resp. $R$) into a disjoint union of three sets $(S_1, S_2, \text{ and } S_{12})$ with the properties that $\mu_1(S_2) = 0 = \mu_2(S_1)$ and that $\mu_1 \approx \mu_2$ on $S_{12}$. Although these properties do not completely determine the three sets, we may uniquely define $\mu_1(\text{supp } \mu_2)$ as $\mu_1(S_{12})$, and $\mu_1((\text{supp } \mu_2))$ as $\mu_1(S_1)$. If $\mu_1$ and $\mu_2$ are probability measures, it follows that $0 \leq \mu_1(\text{supp } \mu_2) \leq 1$, and that $\mu_1 \ll \mu_2 \iff \mu_1(\text{supp } \mu_2) = 1$, while $\mu_1 \perp \mu_2 \iff \mu_1(\text{supp } \mu_2) = 0$.

**Lemma.** Suppose $\mu_1$ and $\mu_2$ are finite Borel measures on $S$ ($= D$ or $R$). If we let $v_1 = \sqrt{\mu_1} \sqrt{\mu_2}$ and $v_n = \sqrt{\mu_1} \sqrt{v_{n-1}}$, then $\lim_{n \to \infty} v_n(S) = \mu_1(\text{supp } \mu_2)$.

This lemma together with a countable number of applications of the theorem yields
COROLLARY 2. If \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) are not mutually singular, then \( \sigma_1((\text{supp } \sigma_2)^c) < \infty \) and \( \tilde{\rho}_1((\text{supp } \tilde{\rho}_2)^c) = \exp(-T\sigma_1((\text{supp } \sigma_2)^c)) \). Thus \( \tilde{\rho}_1 \ll \tilde{\rho}_2 \iff K < \infty \) and \( \sigma_1 \ll \sigma_2 \); and \( \tilde{\rho}_1 \approx \tilde{\rho}_2 \iff K < \infty \) and \( \sigma_1 \approx \sigma_2 \).

The theorem itself is proved in two parts. It is first shown that when \( K < \infty, \sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2} \geq e^{-TK} \tilde{\rho}_3 \). The basic ingredient in this demonstration is the proposition that for finite Borel measures on \( D, \sqrt{\mu_1 * \nu_1} \sqrt{\mu_2 * \nu_2} \geq (\sqrt{\mu_1} \sqrt{\mu_2}) * (\sqrt{\nu_1} \sqrt{\nu_2}) \), where \(*\) denotes convolution. In the second part of the proof, it is first noted that

\[
\int_D d(\sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2}) \leq \left( \int_R d(\sqrt{\rho_1^{*T/n}} \sqrt{\rho_2^{*T/n}}) \right)^n
\]

for all integers \( n \), and then a lengthy determination of the fact that

\[
(2) \lim_{a \to 0} \int_R d(\sqrt{\rho_1^{*a}} \sqrt{\rho_2^{*a}}) - 1 \quad \text{allows us to conclude that } \int_D d(\sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2}) \leq e^{-TK}. \]

The conclusions of the theorem then follow immediately.

Our results in the purely Gaussian case (\( \sigma_1 = \sigma_2 = 0 \)) that \( \tilde{\rho}_1 \approx \tilde{\rho}_2 \) if \( \beta_1 = \beta_2 \) (and \( \alpha_1 = \alpha_2 \) when \( \beta_1 = \beta_2 = 0 \)) and that otherwise \( \tilde{\rho}_1 \perp \tilde{\rho}_2 \) are well known [1], [2]. In the general case, Skorokhod [3, Chapter 4] has previously obtained a (somewhat complicated) set of sufficient conditions for the equivalence of \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) and has calculated the Radon-Nikodym derivative, \( d\tilde{\rho}_1/d\tilde{\rho}_2 \), under those conditions. It can readily be shown that the conditions for equivalence of Corollary 2 actually imply Skorokhod’s conditions which are thus seen to be in fact necessary as well as sufficient.

REMARK 1. When \( \tilde{\rho}_1 \approx \tilde{\rho}_2 \), the results of the theorem, as stated in terms of \( \sqrt{\tilde{\rho}_1} \sqrt{\tilde{\rho}_2} \), can be regarded as a kind of symmetric substitute for a determination of \( d\tilde{\rho}_1/d\tilde{\rho}_2 \) or \( d\tilde{\rho}_2/d\tilde{\rho}_1 \).

REMARK 2. The calculation of (2) for infinitely divisible measures on \( R^s \) has also been carried out, leading to the expected results. The requirement that \( \gamma = 0 \) when \( \beta_1 = \beta_2 = \beta = 0 \) in order to have \( K \) finite is replaced by the condition that \( \tilde{\gamma} \) be orthogonal to the null space of \( B \), and \( \gamma^2/2\beta \) in the definition of \( K \) is replaced by \( (\tilde{\gamma}, (2B)^{-1} \tilde{\gamma}) \). Here, \( \tilde{\gamma} \) and the positive semi-definite matrix \( B \) are respectively the \( R^s \)-analogues for \( \gamma \) and \( \beta \). The \( s \)-dimensional analogues to our main results then follow.

REMARK 3. It is clear that our results can be extended to the measures associated with nonhomogeneous processes with independent increments (on finite or infinite time intervals). Within this more general context, our present results will play a “local” role.

REMARK 4. The measures \( \rho_k^{*t} \), acting by convolution on the bounded continuous functions, define contraction semigroups, \( \exp(tA_k) \); while the
measure $\sqrt{\rho_1^{\#}} \sqrt{\rho_2^{\#}}$ similarly defines a contraction operator which we denote by $F(t)$. $F(t)$ is in general not a semigroup, but when $K < \infty$, the proof of (2) yields the very strong results that

$$\lim_{t \to 0} \frac{F(t) - I}{t} = A_3 - K I \tag{3}$$

and

$$\lim_{n \to \infty} \left( F \left( \frac{t}{n} \right) \right)^n = \exp(t(A_3 - K I)), \tag{4}$$

where $I$ is the identity operator. In (3) the limit is taken strongly (on the domain of $A_3$), while in (4) the limit is taken in the uniform operator topology.

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