DE RHAM’S INTEGRALS AND LEFSCHETZ
FIXED POINT FORMULA FOR $d''$ COHOMOLOGY

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We give here a brief sketch of a different approach to the Atiyah-Bott type Lefschetz fixed point formula for Dolbeault complexes. Our method is based on an extension to the complex case of de Rham’s integral formulas for Kronecker indices [7]. This approach yields results for general fixed point sets, and in particular we shall give here a formula for isolated degenerate fixed points. Details and related results will appear elsewhere.

Following notations in [1], [2], [3], let $X$ be a compact complex analytic manifold of complex dimension $n$,

$$
\Gamma(\Lambda^p.X): 0 \rightarrow \Gamma(\Lambda^{p,0}) \rightarrow \ldots \rightarrow \Gamma(\Lambda^{p,n}) \rightarrow 0,
$$

for $0 \leq p \leq n$, the $p$th Dolbeault complex, $f: X \rightarrow X$ a complex analytic mapping with isolated fixed points, and

$$T_{p,q} = \Lambda^p(d^* f^*) \otimes \Lambda^q(d'' f^*) \circ f^* : \Gamma(\Lambda^{p,q}) \rightarrow \Gamma(\Lambda^{p,q})$$

the induced endomorphisms on the complex. In terms of $T_{p,q}$ we define, as in [3],

$$\text{graph} \{ T_{p,q} \} \in \Gamma'(\Lambda^{p,q} \boxtimes (\Lambda^{p,q}'))$$

where $(\Lambda^{p,*})'$ denotes the geometric dual and $\Gamma'$ the space of distributions. It is then seen that

$$\text{graph} \{ T_p \} = \sum_{q=0}^n \text{graph} \{ T_{p,q} \} \in H'(\Lambda^{p,*} \boxtimes (\Lambda^{p,*}')).$$

Similarly define

$$\Delta_p = \sum_{q=0}^n \text{graph} \{ I_{p,q} \} \in H'((\Lambda^{p,*})' \boxtimes \Lambda^{p,*})$$

where $I_{p,q}: \Gamma((\Lambda^{p,q}')) \rightarrow \Gamma((\Lambda^{p,q}))$ is the identity. Analogous to [3], [6], one deduces from Poincaré duality and Künneth formula that the Lefschetz number

$$L(f^{p,*}) = \sum (-1)^q \text{trace} \{ T_{p,q}^{*} \}$$


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is given by

\[
L(f^p) = (\ast\text{graph}\{T_p, \overline{A}_p\})
\]

where the inner product is defined as usual by \((\alpha, \beta) = \int z \Lambda^* \overline{\beta} \). The product \((\ast\text{graph}\{T_p, \overline{A}_p\})\) is determined at the intersection of singular supports of the two distributions, and may be computed locally. Let a local coordinate map be given, through which the fixed point is mapped to origin in \(C^{2n}\), the piece of singular support of \(\Delta_p\) is mapped to a subset \(V\) of the diagonal, and that of \(\text{graph}\{T_p\}\) is mapped to a set \(U\). Denote by \(\text{graph}\{T_p\}_U, (\Delta_p)_{\nu}\) the distributions transformed to \(C^{2n}\). Since in euclidean space \((d\delta + \delta d)G = 1\) [7], and \(d\delta + \delta d = 2(d'\delta' + \delta'd') = 2(d^n\delta^n + \delta^nd^n)\) we can write

\[
(\ast\text{graph}\{T_p\}_U, \overline{A}_{p\nu}) = 2(d'\delta'^* G \text{ graph}\{T_p\}_U, \overline{A}_{p\nu}) + 2(\ast\text{graph}\{T_p\}_U, \delta'd'G\Delta_{p\nu})
\]

and the r.h.s. is given by integrals of smooth functions. The sum is invariant as we increase the support of \(V\) to the full diagonal \(\Delta\) in \(C^{2n}\), and we find the second term vanishes while the first term becomes

\[
2 \int_{\Delta} \int_{\partial U} \delta_z^* P g(z, \zeta)
\]

where \(g(z, \zeta)\) is the Green’s form in \(C^{2n}\), and

\[P: \Lambda \otimes \Lambda \to \sum_q \Lambda^{n-p,n-q} \otimes \Lambda^{p,q}\]

is the projection determined by \(\Delta_p\). Suppose now the mapping is described locally by

\[z_{n+i} = f_i(z_1, \ldots, z_n), \quad 1 \leq i \leq n,\]

and denote \(h_t(z) = z_i - f_i\). Let \(A_p(z)\) be holomorphic functions defined by

\[\sum A_p(z)t^{n-\rho} = \det\left(tI + \left(\frac{\partial f_i}{\partial z_j}\right)\right).\]

Then (2) is evaluated to be

\[
\frac{(n-1)!}{(2\pi i)^n} \int_{\partial U} A_p(z)
\]

\[
\sum h_j dh_1 \land dz_1 \land \cdots \land dz_{j-1} \land dz_j \land dh_{j+1} \land \cdots \land dh_n \land dz_n, \quad (\sum |h_i|^2)^{\rho}
\]

In the case of a simple fixed point, a change of variable together with Bochner’s integral formula [4] applied to (3) yields the formula (4.9) of [2].
In the case of an isolated nonsimple fixed point, we shall give in a subsequent paper an algorithm for computing (3). It will be seen that in this case, the algorithm gives the same computation as Grothendieck’s residue symbol \([5]\). In the latter’s notation (3) can be written as:

\[
\text{Res} \left[ \frac{A_p(z)dz_1 \wedge \cdots \wedge dz_n}{h_1 \cdots h_n} \right].
\]

A cruder and quite different approach to this problem is given in [8].

REFERENCES


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