A REMARK ON STRONG PSEUDOCONVEXITY FOR ELLIPTIC OPERATORS

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The purpose of this note is to give a kind of intrinsic characterization of second order elliptic operators.

Let \( p(x, D) \) be an \( m \)th order elliptic operator defined on an open subset \( U \) of \( \mathbb{R}^n \). Let \( p_m(x, \xi) \) be its leading symbol. Let \( \varphi \) be a smooth function on \( U \) with the property that \( \text{grad} \, \varphi \neq 0 \) when \( \varphi = 0 \). The hypersurface, \( \varphi = 0 \), is said to be strongly pseudoconvex at a point \( x \in \varphi^{-1}(0) \) if

\[
\sum \frac{\partial^2 \varphi}{\partial x_j \partial x_k} p_m^{(j)}(x, \xi) \overline{p_m^{(k)}(x, \xi)} + \tau^{-1} \text{Im} \sum p_{m,k}(x, \xi) \overline{p^{(k)}(x, \xi)} > 0,
\]

for all \( \xi = \eta + i \tau \text{grad} \, \varphi \), where \( \eta \in \mathbb{R}^n \) and \( 0 \neq \tau \in \mathbb{R} \), satisfying the equations:

\[
p_m(x, \xi) = 0 = \sum p_m^{(j)}(x, \xi) \, \partial \varphi / \partial x_j.
\]

(See Hörmander [2, Chapter 8].)

If \( p \) is a second order operator and its leading symbol is real, then, for \( \eta, N \in \mathbb{R}^n \) not multiples of each other, the equations

\[
p_2(x, \eta + \tau N) = 0 = \sum p_2^{(j)}(x, \eta + \tau N) N_j
\]

have no solutions, so condition (1) is satisfied trivially. This proves

**Proposition 1.** If \( p(x, D) \) is second order and its leading symbol is real, then every hypersurface is strongly pseudoconvex.

In this note we will prove a result in the other direction, namely,

**Proposition 2.** If \( n \geq 3 \) and every hypersurface is strongly pseudoconvex then \( p(x, D) \) is second order.

**Remark 1.** If there exist vectors \( \eta, N \) satisfying (3) it is easy to construct a \( \varphi \), with \( \text{grad} \, \varphi(x) = N \), violating (1). Therefore for every surface with normal, \( N \), at \( x \) to be strongly pseudoconvex at \( x \) it is necessary and sufficient that there be no \( \eta, N \) satisfying (3). Hence Proposition 2 can be reformulated as a simple algebraic assertion, namely,

**Proposition 3.** Let \( p(\zeta), \zeta \in \mathbb{C}^n \) be a homogeneous polynomial of degree \( m \). Assume \( n \geq 3 \), and assume \( p \) satisfies the following conditions:

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432
(a) \( p \) has no real zeros except 0.

(b) For any fixed pair of vectors \( \xi \) and \( \eta \) in \( \mathbb{R}^n \), \( \xi \) and \( \eta \) not multiples of each other, the equation \( p(\xi + \tau \eta) = 0, \tau \in \mathbb{C} \), has simple roots.

Then \( m = 2 \).

REMARK 2. It is easy to see that condition (4) is equivalent to the following condition:

\[
\sum \frac{\partial p(\xi)}{\partial \xi^i} \xi^i \neq 0 \quad \text{when} \quad p(\xi) = 0, \xi \in \mathbb{C}^n \neq 0.
\]

REMARK 3. It is enough to prove Proposition 3 when \( n = 3 \). In fact choose coordinates so that the coefficient of \( \xi^n \) in \( p(\xi) \) is nonzero. If the theorem is true in dimension 3 then \( p(\xi_1, \xi_2, \xi_3, 0, \ldots, 0) \) is of degree 2 so \( m = 2 \).

We will assume from now on that \( n = 3 \) and that \( p(\xi) \) satisfies condition (5). Since \( p(\xi) \) is homogeneous in \( \xi \) its zero variety defines an algebraic curve, \( \gamma \), in \( \mathbb{P}^2(\mathbb{C}) \). By (5), \( \text{grad} \ p(\xi) \neq 0 \) when \( p = 0 \) so this curve is nonsingular.

We identify \( \mathbb{P}^2(\mathbb{C}) \) with the set of one-dimensional subspaces of \( \mathbb{C}^3 \). Given \( l \in \mathbb{P}^2(\mathbb{C}) \), let \( L_l = \{ \xi \in \mathbb{C}^3, \xi \in l \} \) and let \( L \) be the vector bundle on \( \mathbb{P}^2(\mathbb{C}) \) whose fiber at \( L \) is \( L_l \). Let \( L_\gamma = L \mid \gamma \). Given \( l \in \mathbb{P}^2(\mathbb{C}) \), let \( \xi \) be a nonzero vector on \( l \) and let

\[
H_l = \left\{ (a_1, a_2, a_3) \in \mathbb{C}^3, \sum \frac{\partial p(\xi)}{\partial \xi^i} (\xi) a_i = 0 \right\}.
\]

Let \( H \) be the vector bundle on \( \gamma \) whose fiber at \( l \) is \( H_l \). By Euler's identity \( L_\gamma \subset H \). Let \( J = H/L_\gamma \).

LEMMA 1. The Chern class \( c_1(J) = 0 \).

PROOF. Let \( E \) be the trivial bundle over \( \gamma \) with fiber \( \mathbb{C}^3 \). Let \( L_\xi \), be the line bundle over \( \gamma \) whose fiber at \( l \in \mathbb{P}^2(\mathbb{C}) \) is \( \{ \xi, \xi \in l \} \). It is easy to see that \( L \) is the dual bundle of \( L \) so \( c_1(L_\xi) = -c_1(L) \). The composite bundle map \( L \to E \to E/H \) is bijective by condition (5) so \( c_1(E/H) = c_1(L) \). Since \( E \) is trivial, \( 0 = c_1(E) = c_1(L) + c_1(J) + c_1(E/H) = c_1(L) + c_1(J) - c_1(L) \) = \( c_1(J) \). Q.E.D.

Let \( T \) be the holomorphic tangent bundle of \( \gamma \). There is a canonical identification \( T \cong \text{Hom}(L_\gamma, J) \), so \( c_1(T) = c_1(J) - c_1(L_\gamma) = -c_1(L_\gamma) \). By Riemann-Roch, \( c_1(T) = 2 - 2g \), \( g \) being the genus of \( \gamma \) (see Gunning [1, p. 110]) so \( c_1(L_\gamma) = 2g - 2 \). On the other hand \( c_1(L_\gamma) = -m \).

(Proof) Let \( \omega \) be a linear functional on \( \mathbb{C}^3 \). Then \( \omega \) defines a section of \( L^* \) whose zero set is the hyperplane \( \omega = 0 \). The Chern class of \( L^* \mid \gamma \) is the number of points in which this hyperplane intersects \( \gamma \), which is just the
degree of \( p \) for hyperplanes in general position.)

To summarize we have proved the equality

\[
2g - 2 = -m.
\]

Since \( m \) is positive this can be satisfied only for \( g = 0 \) and \( m = 2 \). Q.E.D.

**BIBLIOGRAPHY**


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