A RADIAL AVERAGING TRANSFORMATION, CAPACITY AND CONFORMAL RADIUS

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Communicated by F. W. Gehring, October 18, 1971

Introduction. Let \( \mathcal{D} = \{D_1, \ldots, D_n\} \) be a family of domains in the plane, containing the origin. We define a radial averaging transformation \( \mathcal{R}_A \) on \( \mathcal{D} \) by which we obtain a starlike domain \( D^* \). When \( \mathcal{D} \) is such that the domains \( D_1, \ldots, D_n \) are obtained from a fixed domain \( D \) by rotation or reflexion, \( \mathcal{R}_A \) becomes a radial symmetrization. One of the results we present is an inequality relating the conformal radius of \( D^* \) to the conformal radii of \( D_1, \ldots, D_n \) at the origin. This result includes, as particular cases, the radial symmetrization results of Szegö [6] (for starlike domains), Marcus [4] (for general domains) and Aharonov and Kirwan [1]. The inequality for the conformal radii is obtained via an inequality for conformal capacities. A number of applications in the theory of functions is described.

1. Let \( M \) be the half strip \( \{(x, y)|0 < x < 1, 0 < y\} \). We shall say that a function \( f \) is of class \( B(M) \) if
   (i) \( f \) is continuous in \( \overline{M} \) (= closure of \( M \));
   (ii) \( 0 < f \leq 1 \) in \( M \);
   (iii) the set \( \Omega_1 = \{(x, y)|f(x, y) < 1\} \cap M \) is bounded;
   (iv) on any half line \( \{x = x_0\} \cap \overline{M}, f \) assumes every value \( \lambda, 0 < \lambda < 1 \), at least once, but not more than a finite number of times;
   (v) \( f \in C^1(\Omega(f)) \), where \( \Omega(f) = \{(x, y)|0 < f(x, y) < 1\} \cap M \);
   (vi) for any line \( x = x_0, 0 \leq x_0 \leq 1 \), the set of points on \( \{x = x_0\} \cap \Omega(f) \) where \( \partial f/\partial y = 0 \) is finite.

If \( f \in B(M) \) we denote

\[ \Omega_0(f) = \{(x, y)|f(x, y) = 0\} \cap M. \]

\[ l(x_0, \lambda; f) = \text{meas}\{x = x_0\} \cap \Omega_\lambda(f) \] (0 \leq \lambda \leq 1),

where the measure is the linear Lebesgue measure. We note that \( l(x_0, \lambda; f) \) is a strictly monotonic increasing function of \( \lambda, 0 \leq \lambda \leq 1 \).

We now introduce

**Definition 1.1.** Let \( \mathcal{F} = \{f_1, \ldots, f_n\} \subset B(M) \) and let \( A = \{a_1, \ldots, a_n\} \) be a set of positive numbers such that \( \sum_{j=1}^n a_j = 1 \). Denote

\[ AMS 1970 \text{ subject classifications. Primary 30A44, 30A36; Secondary 31A15, 30A32.} \]

\[ Key \text{ words and phrases. Conformal capacity, Dirichlet integral, conformal radius, radial symmetry, complex analytic functions in the unit disk.} \]
(1.3) \[ l^*(x, \lambda) = \sum_{j=1}^{n} a_j l(x, \lambda; f_j); \]

\[ \Omega^*_\alpha = \Omega^*_{\alpha}(\mathcal{F}, A) = \{(x, y) | 0 < y < l^*(x, \lambda) \} \cap M \quad (0 < \lambda \leq 1), \]

(1.4) \[ \Omega^*_\alpha = \Omega^*_{\alpha}(\mathcal{F}, A) = \{(x, y) | 0 \leq y \leq l^*(x, 0) \} \cap M, \]

\[ \Omega^* = \Omega^*(\mathcal{F}, A) = \Omega^*_{\alpha} - \Omega^*_{\alpha}. \]

Then the linear averaging transformation \( \mathcal{L}_A \) on \( \mathcal{F} \) is defined as follows:

\[ f^*(x, y) = \mathcal{L}_A(\mathcal{F}) = 0, \text{ if } (x, y) \in \Omega^*_a, \]

(1.5) \[ f^*(x, y) = \lambda, \text{ if } y = l^*(x, \lambda), 0 < \lambda < 1, \]

\[ = 1, \text{ if } (x, y) \in M - \Omega^*_a. \]

The following two results are the main steps in the derivation of the main theorems.

**Lemma 1.1.** Let \( \mathcal{F} \) and \( A \) be as in Definition 1.1. Then \( f^* \) is uniformly Lipschitz in \( M \).

**Theorem 1.1.** Let \( \mathcal{F} \) and \( A \) be as in Definition 1.1. Let \( G(t) \) be a function defined for \( t \geq 0 \) such that \( G(t) \) is continuous, convex and nondecreasing. Then, with the notations introduced above, we have

(1.6) \[ \int \int_{\Omega^*_a} G(1 + |\nabla f^*|^2)^{1/2} \, dx \, dy \leq \sum_{j=1}^{n} a_j \int \int_{\Omega(f_j)} G((1 + |\nabla f_j|^2)^{1/2}) \, dx \, dy, \]

where \( \Omega(f_j) = \Omega_{1}(f_j) - \Omega_0(f_j) \).

**Corollary.**

(1.7) \[ \int \int_{\Omega^*_a} |\nabla f^*|^p \, dx \, dy \leq \sum_{j=1}^{n} a_j \int \int_{\Omega(f_j)} |\nabla f_j|^p \, dx \, dy \quad (1 \leq p). \]

Note that the left side of (1.6) is meaningful because of Lemma 1.1.

2. A condenser in the plane is a system \( C = (\Omega, E_0, E_1) \) where \( \Omega \) is a domain and \( E_0, E_1 \) are disjoint closed sets whose union is the complement of \( \Omega \). We shall assume also that \( E_0 \) is compact and \( E_1 \) unbounded. An alternative notation for \( C \) will be \( C = (D, E_0) \) where \( D = \Omega \cup E_0 \).

If \( \Omega \) satisfies the segment property (i.e., for any point \( P \) on the boundary of \( \Omega \) there exists a segment \( PP' \) lying outside \( \Omega \)), there exists a unique function \( \omega \), called the potential function of \( C \), such that \( \omega \) is harmonic in \( \Omega \) and continuous in the extended plane and such that \( \omega \equiv 0 \) on \( E_0 \) and \( \omega \equiv 1 \) on \( E_1 \). In this case the conformal capacity of \( C \) may be defined by
We now introduce

**DEFINITION 2.1.** Let \( \mathcal{D} = \{D_1, \ldots, D_n\} \) be a family of open sets in the complex plane \( z \). Suppose that the closed disk \( |z - z_0| \leq \rho \) (for some \( \rho > 0 \)) is contained in \( \bigcap_{j=1}^{n} D_j \). Let

\[
K_j^p(\phi) = \{r|z = z_0 + re^{i\phi} \in D_j, \rho < r < \infty\} \quad (0 \leq \phi < 2\pi);
\]

\[
l_j^p(\phi) = \int_{K_j^p(\phi)} \frac{dr}{r} \quad \text{and} \quad R_j^p(\phi) = R(\phi; D_j; z_0) = \rho \exp l_j^p(\phi).
\]

(Note that \( R_j^p(\phi) \) does not depend on \( \rho \).)

Let \( \mathbf{A} = \{a_1, \ldots, a_n\} \) be a set of positive numbers such that \( \sum_{j=1}^{n} a_j = 1 \). Set

\[
R^p(\phi) = \prod_{j=1}^{n} R_j^p(\phi)^{a_j};
\]

\[
D^* = \mathcal{R}_A(\mathcal{D}; z_0) = \{z = z_0 + re^{i\phi} | 0 \leq r < R^p(\phi), 0 \leq \phi < 2\pi\}.
\]

We shall say that \( \mathcal{R}_A \) is a radial averaging transformation on \( \mathcal{D} \) with center \( z_0 \).

If \( \{C_j\}_{j=1}^{n} \) is a family of condensers, \( C_j = (\Omega_j, E_{0,j}, E_{1,j}) = (D_j, E_{0,j}) \) where \( \bigcap_{j=1}^{n} E_{0,j} \ni \{z - z_0| \geq \rho\} \) we define

\[
C^* = \mathcal{R}_A(C_j; z_0) = (D^*, E_0^*)
\]

where \( D^* = \mathcal{R}_A(D_j; z_0) \) and \( E_0^* = \mathcal{R}_A(E_{0,j}; z_0) \). \( E_0^* \) is defined in the same way as \( D^* \) except that in (2.5) we have \( 0 \leq r \leq R^p(\phi) \).

We can now formulate the main result.

**THEOREM 2.1.** Let \( \{C_1, \ldots, C_n\} \) be a family of condensers as above, and let \( C^* \) be defined as in (2.6). Suppose that the domains \( \Omega_1, \ldots, \Omega_n \) have the segment property. Then

\[
I(C^*) \leq \sum_{1}^{n} a_j I(C_j).
\]

The proof is based on Theorem 1.1. We may assume that \( z_0 = 0 \) and \( \rho = 1 \). We map the domain \( |z| < 1 \), cut along the positive real axis, by \( w = \ln z \) onto the half strip \( 0 < u < \infty, 0 < v < 2\pi (w = u + iv) \). Let \( \omega_j \) be the potential function of \( C_j \). Denote by \( f_j(u, v) \) the function \( \omega_j \) represented in \( (u, v) \) coordinates. Then we apply Theorem 1.1 (or, more precisely, inequality (1.7) with \( p = 2 \)) to \( \mathcal{F} = \{f_1, \ldots, f_n\} \) in the strip mentioned above.
If \( D \) is a domain in the plane and \( z_0 \in D \), denote by \( r(z_0, D) \) the conformal (or inner) radius of \( D \) at \( z_0 \). (For definition and properties see for instance Hayman [3, pp. 78–83].) Using a theorem of Pólya and Szegő [5] on the relation between conformal radius and conformal capacity and Theorem 2.1 we obtain

**Theorem 2.2.** Let \( \mathcal{D} = \{D_1, \ldots, D_n\} \) be a family of domains such that \( z_0 \in \bigcap_1^n D_j \). Let \( D^* = R_A(\mathcal{D}; z_0) \) (Definition 2.1). Then

\[
\prod_{j=1}^n r(z_0, D_j)^{\alpha_j} \leq r(z_0, D^*).
\]

As a first application of Theorem 2.2 we obtain the following symmetrization result:

**Theorem 2.3.** Let \( f(z) = a_1 z + a_2 z^2 + \cdots \) be an analytic function in the unit disk \( |z| < 1 \). Let \( D \) be the image of \( |z| < 1 \) by \( w = f(z) \). Let \( A = \{a_j\}_{j=1}^n \) be a set of positive numbers such that \( \sum_1^n a_j = 1 \), let \( \{x_j\}_{j=1}^n \) be a set of integers \( (x_j \neq 0) \) and let \( \{\beta_j\}_{j=1}^n \) be an arbitrary set of real numbers.

If \( R(\phi) = R(\phi; D; 0) \) (see (2.3)) set

\[
R^*(\phi) = \prod_{j=1}^n R(x_j \phi + \beta_j)^{b_j}, \quad \text{where } b_j = a_j/|x_j|; \\
D^* = \{w = \sigma e^{i\phi} | 0 \leq \sigma < R^*(\phi), 0 \leq \phi < 2\pi\}.
\]

Then

\[
|a_1| \leq r(0, D) \leq r(0, D^*)^{1/b}, \quad \text{where } b = \sum_1^n b_j.
\]

Theorem 2.3 includes as particular cases the radial symmetrization results of Szegő [6], Marcus [4] and Aharonov and Kirwan [1].

We bring now two applications of the preceding theorems.

**Theorem 2.4.** Let \( f(z) \) and \( D \) be as in Theorem 2.3. Denote

\[
D_t = \{w = \sigma e^{it\phi} | 0 \leq \sigma < R(\phi)^t, 0 \leq \phi < 2\pi\}, (0 < t < 1),
\]

where \( R(\phi) = R(\phi; D; 0) \). Then

\[
|a_1| \leq r(0, D) \leq r(0, D_t)^{1/t}.
\]

**Theorem 2.5.** Let \( f(z) = z + a_2 z^2 + \cdots \) and \( D \) be as in Theorem 2.3. Let \( R^*(\phi) \) be defined as in (2.9). Suppose that \( R^*(\phi) \leq M \leq \infty \), \( 0 \leq \phi < 2\pi \). Suppose also that for some set of \( m \) rays issuing from the origin, with arguments \( \phi_1, \ldots, \phi_m \) we have

\[
\sup_{1 \leq j \leq m} R^*(\phi_j) = K.
\]
Let $D_0$ be the disk $|w| < M$ (the entire plane if $M = \infty$) cut along the rays $w = \sigma e^{j\phi}, K_0 \leq \sigma < M, j = 1, \ldots, m$, where $K_0$ is so chosen that $r(0, D_0) = 1$. (It follows from our assumptions that $M \geq 1$.) Then $K_0 \leq K$.

Theorem 2.5 implies a number of special “covering” theorems such as Theorem 5 and 6 of Marcus [4] and Theorem 4.2 of Aharonov and Kirwan [1].

A complete presentation of the results described in this note and additional applications will appear elsewhere. We mention also that a discussion of radial averaging transformations with metrics of the form $g(r) \, dr \, d\phi$ is given in [2].

REFERENCES