CONSTRUCTING ISOTOPIES IN NONCOMPACT 3-MANIFOLDS

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Introduction. Let $M$ be a noncompact, orientable 3-manifold with a (possibly empty) boundary $\partial M$. Suppose $g$ and $h$ are homeomorphisms of $M$ onto itself. When is $g$ isotopic to $h$? This question was essentially answered in the compact case by Waldhausen in [3]; roughly the answer given was—when $g$ is homotopic to $h$. We will show that essentially the same answer can be given for a large and interesting class of noncompact manifolds; these manifolds include Whitehead-type contractible open subsets of $R^3$. Full proofs of the theorems stated below will be given elsewhere.

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Preliminaries. The ambient manifolds considered here are orientable, triangulable and 3-dimensional. By a surface in $M$, we mean a 2-dimensional, triangulable manifold which is properly imbedded in $M$. (Everything is considered from the piecewise linear point of view.) $M$ is an irreducible manifold if every 2-sphere in $M$ bounds a ball in $M$. For noncompact manifolds this implies that $M$ is aspherical. A surface $F$ in $M$ or $\partial M$ different from a 2-sphere is incompressible in $M$ if $\pi_1(F) \to \pi_1(M)$ is a monomorphism. $M$ is boundary-irreducible if each component of $\partial M$ is an incompressible surface. Finally we need the notion of a hierarchy for a manifold. The triple $(F_j, U(F_j), M_j), j = 1, 2, \ldots$, is a hierarchy for $M = M_1$ if each $F_j$ is a compact incompressible orientable surface in $M_j$, $M_{j+1} = \text{cl}(M_j - U(F_j))$, where $U(F_j)$ is a regular neighborhood of $F_j$ in $M_j$ [4], and $M - \bigcup_j U(F_j)$ is a collection of balls. If $M$ is compact we require the sequence $F_j$ to be finite. For $M$ compact these surfaces have been constructed by Haken when $M$ is irreducible and has an incompressible surface. Waldhausen uses the hierarchy to prove the isotopy theorem in the compact case.


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Noncompact manifolds admitting a hierarchy are called *end-irreducible*; they have been introduced by E. M. Brown, for manifolds of any dimension using his notions of proper fundamental groups. It can be seen from [1] that the name end-irreducible is appropriate in that it generalizes the idea of boundary-irreducible to ends of a manifold. Moreover any irreducible manifold which is obtained from a compact 3-manifold by removing some incompressible boundary components is an end-irreducible manifold.

**Results for irreducible, end-irreducible manifolds.** If the manifold $M$ has vacuous boundary, our result asserts that any orientation-preserving homeomorphism of an irreducible and end-irreducible manifold which is *homotopic* to the identity homeomorphism is *isotopic* to the identity. Notice that since $M$ is aspherical we are saying that any orientation-preserving homeomorphism which induces the “identity” map on $\pi_1(M)$ is isotopic to the identity homeomorphism. More precisely we prove the following two isotopy theorems.

**Theorem 1.** Let $M$ be an irreducible, end-irreducible manifold and $H: (M \times I, \partial M \times I) \to (M, \partial M)$ be a homotopy of an orientation-preserving homeomorphism $h$ to the identity. Then $h$ is isotopic to the identity. [If $H|\partial M \times I$ is already the constant homotopy, the isotopy of $h$ to the identity may be chosen fixed on $\partial M$.]

**Theorem 2.** Let $M$ be an irreducible, end-irreducible, and boundary-irreducible manifold. Assume $\partial M \neq \emptyset$. Suppose $H:M \times I \to M$ is a homotopy of an orientation-preserving homeomorphism $h$ to the identity and suppose that $H$ is a proper map when restricted to each component of $\partial M \times I$. Then $h$ is isotopic to the identity.

(By a *proper map* we mean that the inverse image of compact sets are compact.) In the above theorems we only use the fact that $h$ is orientation-preserving for manifolds that

(a) are bundles with fiber $R$ over closed surfaces, (in Theorem 1),
(b) are bundles with fiber $I$ over open surfaces, (in Theorem 2) or
(c) when some component of $\partial M$ is a plane or an open annulus, (in Theorem 1).

To see that in Theorem 2 it is necessary to assume $H$ is a proper map when restricted to components of $\partial M \times I$ we look at the following example. Let $M$ be a solid torus with two disjoint longitudinal curves removed from $\partial M$. Then $\partial M$ consists of two open annuli. Let $h$ be the homeomorphism of $M$ which rotates the torus so that the boundary components are interchanged. Then $h$ is homotopic to the identity but not isotopic to the identity.

Other examples of end-irreducible manifolds will be given in the next section.
An outline of the proof of Theorem 1 for a manifold $M$ with vacuous boundary. In [1], it is shown that $M$ has an exhausting sequence $\{C_n\}$ of submanifolds with the following properties:

1. $C_n$ is a compact, connected manifold,
2. $C_n \subseteq \mathcal{C}_{n+1}$ and $\bigcup_n C_n = M$,
3. components of $\partial C_n$ are incompressible.

Now one can show that $\{C_n\}$ may be chosen so that

4. $H(C_n \times I) \subseteq \mathcal{C}_{n+1}$.

We show by an inductive procedure that if $h|C_n$ is the identity and $H|C_n \times I$ is the constant homotopy, then there is an isotopy of $h$ fixed on $C_n$ so that the new homeomorphism (which we will still call $h$) is the identity on $C_{n+1}$. Moreover we show that one can change the homotopy to be constant on $C_{n+1}$. Let $F_1, \ldots, F_k$ be the components of $\partial C_{n+1}$. We first show that since $F_1$ and $h(F_1)$ are incompressible surfaces which are homotopic via $H$ in $C_{n+2}$, there is an isotopy of $M$ fixed on $C_n$ which carries $h(F_1)$ onto $F_1$. (We use here that if $M$ is a product bundle, then $h$ is assumed orientation-preserving.) Changing $h$ by this isotopy—still call the homeomorphism $h$—we now have $h(F_1) = F_1$.

The fact that this isotopy can be chosen fixed on $C_n$ rests heavily on the fact that $H|C_n \times I$ is already constant. Thus we now try to change $H|F_1 \times I$ by a homotopy to the constant homotopy. First we attempt to homotope $H|F_1 \times I$ rel $F_1 \times \partial I$ to a map into $F_1$. We show the only difficulties arise when $M$ is a bundle over a closed, nonorientable manifold. Here we again use the assumption that $h$ is orientation-preserving. Next we show that unless $F_1$ is a torus, $H|F_1 \times I$ is homotopic rel $F_1 \times \partial I$ to the constant homotopy. If $F_1$ is a torus it may be necessary to change $h|F_1$ by an isotopy before one can homotope $H|F_1 \times I$ to the desired state. We continue—by induction on the number of components in $\partial C_{n+1}$—to change $h$ by an isotopy fixed on $C_n$ and to change $H|\partial C_{n+1} \times I$ by a homotopy so that in the end, $h|\partial C_{n+1}$ is the identity and $H|\partial C_{n+1} \times I$ is the constant homotopy.

Since $\text{cl}(C_{n+1} - C_n)$ is a manifold with boundary, it admits a hierarchy $G_1, \ldots, G_r$ (see [2]). Again we inductively isotope $h$ and homotope $H$ without disturbing our previous changes so that the end result is that $h$ is the identity on all the $G_i$. Hence $h$ is the identity on $C_{n+1}$ except for a collection of 3-cells. Using Alexander’s Theorem, we conclude $h$ is isotopic to the identity on $C_{n+1}$; using the fact that $M$ is aspherical we have no obstructions to homotoping $H$ to the constant homotopy. This concludes the argument.

Results for irreducible, eventually end-irreducible manifolds. $M$ is eventually end-irreducible if there is a compact subset $C$ of $M$ such that $M - C$ is end-irreducible, or equivalently that $M$ eventually has a hierarchy,
i.e., $M - C$ has a hierarchy. For the isotopy results for such manifolds the following example shows that it is necessary that we assume the homeomorphism $h$ is proper homotopic to the identity. Let $T_0$ be a solid torus linked in the solid torus $T_1$. (See Figure below.)

Let $h : R^3 \rightarrow R^3$ be a homeomorphism of $R^3$ such that $h(T_0) = T_1$. Then $W = \bigcup h'(T_0)$ is the contractible open subspace of $R^3$ described by Whitehead in [5]. By results in [6], one can show that $W$ is an eventually end-irreducible manifold. The homeomorphism $h$ maps $W$ onto itself. Moreover since $W$ is contractible $h$ is homotopic to the identity. Again the results in [6] show that $h$ is not proper homotopic to the identity and hence it certainly is not isotopic to the identity.

**Theorem 3.** Let $M$ be an irreducible, eventually end-irreducible manifold and $H : (M \times I, \partial M \times I) \rightarrow (M, \partial M)$ a proper homotopy of a homeomorphism $h$ to the identity. Then $h$ is isotopic to the identity. If $H|\partial M \times I$ is the constant homotopy, then the isotopy of $h$ to the identity may be chosen fixed on $\partial M$.

**Bibliography**