NONANALYTIC-HYPOELLIPTICITY FOR SOME DEGENERATE ELLIPTIC OPERATORS

BY M. S. BAOUENDI AND C. GOULAOUIC

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We give here an example, as simple as possible, of a degenerate elliptic operator \( \sum_{j=1}^{r} X_j \) where \( X_1, X_2, \ldots, X_r \) are \( r \) vector fields with analytic coefficients which, with their commutators of order 1, span the whole space, and such that there exists a nonanalytic function \( u \) in the Gevrey class \( G_2 \) with \( \sum_{j=1}^{r} X_j^2 u = 0 \).

1. We consider an operator

\[
A = yP + Q
\]

where \( P \) is a second order elliptic (nondegenerate) operator and \( Q \) is a first order operator; we assume the coefficients of \( P \) and \( Q \) are analytic in some neighborhood \( \mathcal{O} \) of the origin in \( \mathbb{R}^n \sim \mathbb{R}^r \times \mathbb{R}^+ \) which is relatively compact in \( \mathcal{O} \). For simplicity we suppose \([P, Q] = PQ - QP = 0\) (however it is possible to consider more general situations). We assume \( n > 1 \).

We obtain the following result:

**Proposition 1.** Let \( V \) be a neighborhood of the origin in \( \mathbb{R}^{n-1} \times \mathbb{R}^+ \) which is relatively compact in \( \mathcal{O} \). There exists a function \( u \in G_2(V) \), whose restriction to any neighborhood of the origin is nonanalytic, such that there exists a constant \( C > 0 \) with

\[
||D^a A^k u||_{L^2(V)} \leq C^{a+k+1} (2k)! (2\alpha)! 
\]

for each \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \).

**Proof.** We note \( \Gamma = \overline{\mathcal{O}} \cap \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}; y = 0 \} \). Let \( g \) be in \( G_2(\Gamma) \) and nonanalytic in any neighborhood of the origin in \( \mathbb{R}^{n-1} \). We construct a function \( u \) in some neighborhood of \( V \) in \( \mathbb{R}^{n-1} \times \mathbb{R}^+ \) such that
by solving a Dirichlet problem in some neighborhood of $V$ in $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Then $u \in G_2(\overline{V})$ and is nonanalytic in any neighborhood of the origin (see [4]).

We get obviously

$$ A^k u = Q^k u \quad \text{in } V. $$

Now the proof can be completed using the following result:

For each $v \in G_2(\overline{V})$, there exists a constant $C > 0$ such that, for every $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$,

$$ \|D^2 Q^k v\|_{L^2(V)} \leq C^{\alpha+1+2k}(2\alpha!)(2k)!. $$

2. We consider the operator

$$ B = A + D_t^2 = yP + Q + D_t^2, $$

in the neighborhood $\mathcal{O} \times \mathcal{R}$ in $\mathbb{R}^{n+1} = \{(x, y, t); x \in \mathbb{R}^{n-1}, y \in \mathcal{R}, t \in \mathcal{R}\}$.

We have the following result:

**Proposition 2.** There exists a neighborhood $W$ of the origin in $\mathbb{R}^{n-1} \times \mathcal{R}_+ \times \mathcal{R}$ and a function $w \in G_2(\overline{W})$ whose restriction to any neighborhood of the origin is not analytic, such that

$$ Bw = 0 \quad \text{in } W. $$

**Proof.** Let us consider the series

$$ w(x, y, t) = \sum_{m=0}^{\infty} t^{2m} \frac{A^m u(x, y)}{(2m)!}. $$

We denote by $D_t$ the operator $-i\partial_t$.

Such a series is also used in [4].
where \( u \) is given by Proposition 1. By using (2) it is easily seen that the function \( w \) is defined in \( W = V \times [-\delta, +\delta] \) where \( \delta \) is some suitable strictly positive number, and satisfies
\[
Bw = 0 \quad \text{in } W
\]
and there exists \( M > 0 \) such that
\[
\|D_{x,y}^k D_t^l w\|_{L^2(W)} \leq M|\alpha|^{n+k+1} k!(2\alpha)!
\]
for each \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \).
Furthermore we have
\[
w(x, y, 0) = u(x, y);
\]
then \( w \) is nonanalytic in any neighborhood of the origin.

3. **Examples and applications.** Let us consider, for example, the following simple case (with \( n = 2 \)):
\[
P = D_x^2 + 4D_y^2,
\]
\[
Q = -2miD_y \quad \text{with } m \text{ integer } \geq 1.
\]
Then
\[
(5) \quad B = y(D_x^2 + 4D_y^2) - 2miD_y + D_t^2.
\]
We use the change of variables
\[
y = z_1^2 + \cdots + z_m^2.
\]
We denote \( \tilde{w} \) by
\[
\tilde{w}(x, z_1, \ldots, z_m, t) = w(x, z_1^2 + \cdots + z_m^2, t)
\]
where \( w \) is given by Proposition 2.

The function \( \tilde{w} \) is in the Gevrey class of order 2 in some neighborhood of the origin in \( \mathbb{R}^{n+2} \) and nonanalytic. (If \( \tilde{w} \) were analytic, the function \( (x, z_1, t) \mapsto w(x, z_1^2, t) \) would be analytic too in some neighborhood of the origin in \( \mathbb{R}^3 \). The latter function is even with respect to \( z_1 \), so the function \( w \) would be also analytic in some neighborhood of the origin in \( \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \), which contradicts Proposition 2.)

By the change of variables (6), the operator \( B \) defined by (5) becomes
\[
H = (z_1^2 + \cdots + z_m^2)D_x^2 + D_{z_1}^2 + \cdots + D_{z_m}^2 + D_t^2
\]
which can be written also in the form
\[
(7) \quad H = \sum_{j=1}^m (z_j D_x)^2 + \sum_{j=1}^m D_{z_j}^2 + D_t^2.
\]
In some neighborhood of the origin in $\mathbb{R}^{m+2}$ we have $H\tilde{w} = 0$. Therefore the following result is proved:

**Theorem.** Let $m$ be an integer $\geq 1$. The following operator

$$H = \sum_{j=1}^{m} (z_j Dx_j)^2 + \sum_{j=1}^{m} D^2_j + D^2_i$$

is not analytic-hypoelliptic in $\mathbb{R}^{m+2}$. More precisely, one can find a function $\tilde{w}$ defined in some neighborhood of the origin, belonging to the Gevrey class of order 2, nonanalytic and such that $H\tilde{w} = 0$.

In fact, we can construct, by the same method used here, a function $\tilde{w}$ which does not belong to any Gevrey class of order $\varepsilon < 2$ and which satisfies $H\tilde{w} = 0$.

The operator $H$ is obviously of the form $\sum X_j^2$ and satisfies the Hörmander condition (see [3]), namely in this case the vector fields $X_j$ and their commutators of order 1 span the whole space.

If, in the example (7), we take $m = 1$, it turns out that the operator $z^2 D_x^2 + D^2_z + D^2_i$ is not analytic-hypoelliptic in $\mathbb{R}^3$; but it is known (see [5]) that the operator

$$z^2 D_x^2 + D^2_z$$

is analytic-hypoelliptic in $\mathbb{R}^2$. Let us point out that M. Derridj and C. Zuily have also announced recently the analytic-hypoellipticity for some classes of operators which can be considered as generalizations of (8).

On the other hand, Proposition 2 gives a negative result of analyticity up to the boundary; positive results were given in [1], [2] for some classes of degenerate elliptic operators apparently not far from those of Propositions 1 and 2.

**Bibliography**


**Department of Mathematics, Purdue University, Lafayette, Indiana 47907**

**Department of Mathematics, University of Paris XI, 91 Orsay, France**