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NONCOBORDANT FOLIATIONS OF $S^3$

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In this note, we will sketch the construction of uncountably many noncobordant foliations of $S^3$, and a surjective homomorphism $\pi_3(B\Gamma^r_1) \to \mathbb{R}$ [2 \leq r \leq \infty], where $B\Gamma^r_1$ is the classifying space for singular $C^r$ codimension one foliations constructed by Haefliger ([3], [4]).

Godbillon and Vey have recently discovered certain cohomology classes associated with foliations, and more generally, with $\Gamma^r_p$-structures or singular $C^r$ codimension $p$ foliations [2]. The cohomology invariant $\Gamma_F$ is defined very simply for a codimension one, transversely oriented foliation $F$ determined by a $C^2$ one-form $\omega$. The condition of integrability for a one-form is $d\omega \wedge \omega = 0$. Then for some one-form $\theta$, $d\omega = -\theta \wedge \omega$. $\Gamma_F$ is defined to be the deRham cohomology class of the closed form $\theta \wedge d\theta$. If $F$ is not transversely oriented, $\Gamma_F$ may be defined via two-sheeted covers. If $F$ is $C^2$ but not given by a $C^2$ one-form, it is still possible to define $\Gamma_F$. $\Gamma_F$ depends only on $F$, not on $\omega$ and $\theta$, and is natural; so if $f: M \to N$, where $N$ has foliation $F$ and $f$ induces a foliation $f^*F$ on $M$, then $\Gamma_{f^*F} = f^*\Gamma_F$. It follows that $\Gamma_F[M^3]$ is an invariant of the cobordism class of $F$. That is, if $\partial N^4 = M_1 + -M_2$, and if $N^4$ has foliation $F$ transverse to $\partial N^4$ inducing $F_1$ on $M_1$ and $F_2$ on $M_2$, then $\Gamma_{F_1}[M_1] = \Gamma_{F_2}[M_2]$.

The form $\theta \wedge d\theta$ may be interpreted as a measure of the helical wobble of the leaves of $F$, as in Figure 1. In order that the cohomology class $\Gamma_F$ be nontrivial, there must be some kind of global phenomenon corresponding to helical wobble.

Now consider the hyperbolic plane $H^2$ and its unit tangent bundle $T_1(H^2)$. There is a foliation $F$ of $T_1(H^2)$ invariant under the isometries of $H^2$: each leaf of $F$ consists of the forward unit tangents to a family of parallel geodesics. In non-Euclidean geometry, parallel means asymptotic
Let $P$ be any convex polygon in $H^2$. We will construct a foliation of the three-sphere $S^3$ depending on $P$. Let the sides of $P$ be labelled $s_1, \ldots, s_k$, and let the angles have magnitudes $\alpha_1, \ldots, \alpha_k$. Let $Q$ be the closed region bounded by $P \cup P'$, where $P'$ is the reflection of $P$ through $s_1$. Let $Q_\varepsilon$ be $Q$ minus an open $\varepsilon$-disk about each vertex. If $p: T_1(H^2) \to H^2$ is projection, then $p^{-1}(Q)$ is a solid torus (with edges) with foliation $F_1$ induced from $F$. For each $i$, there is a unique orientation-preserving isometry of $H^2$, denoted $f_i$, which matches $s_i$ point-for-point with its reflected image $s_i'$. We glue the cylinder $p^{-1}(s_i \cap Q_\varepsilon)$ to the cylinder $p^{-1}(s_i' \cap Q_\varepsilon)$ by the differential $dl_i$ for each $i > 1$, to obtain a manifold $M = (S^2 \text{ minus } k \text{ punctures}) \times S^1$, and a glued foliation $F_2$ induced from $F_1$. With a little thought, we see that $F_2$ intersects each boundary component of $M$, $T_1^2$, transversely, and induces there a foliation analytically conjugate to a linear foliation of slope $\alpha_k/\pi$. Now we spin each of these linear foliations around a torus leaf; then we glue in $k$ Reeb components, the first one sideways so that we obtain $S^3$ with a smooth foliation $F_P$. 

**Figure 1. Helical wobble of the leaves of a foliation.** The disks here have a constant angle of inclination to the central axis, in a direction which rotates at a constant rate. Their centers are evenly spaced. The arrows are perpendicular to the central axis, and they represent the direction in which the leaves are most squeezed together.
THEOREM. $\Gamma_{F_p}[S^3] = 4\pi \text{Area}(P)$.

INDICATION OF PROOF. When the forms $\omega$ and $\theta$ defining $F$ on $T_1(H^2)$ are invariant under the orientation-preserving isometries, $\theta \wedge d\theta$ is the volume form. The foliation $F_p$ does not depend on $\varepsilon$; if $\varepsilon$ is small, and if $\omega$ and $\theta$ defining $F_p$ agree with the forms on $T_1(H^2)$ except in a small neighborhood of the Reeb components, they may be extended so the integral of $\theta \wedge d\theta$ is nearly 0 over the Reeb component neighborhoods. Then $\Gamma_{F_p}[S^3]$ approximates, therefore equals, the volume of $p^{-1}(Q)$.

COROLLARY. There are uncountably many noncobordant $C^\infty$ foliations of $S^3$. The Godbillon-Vey invariant is a surjective homomorphism of $\pi_3(B\Gamma_1')$ $[2 \leq r \leq \infty]$ onto the reals.

There are further examples of foliations which are analytic, transverse to the fibres of an $S^1$ bundle over a surface, and for which $\Gamma_F[M]$ takes all real values. From this follows the

THEOREM. $H_3(B\Gamma_1'; Z)$ and $H_2(\text{Diff}^r_+(S^1); Z)$ $[2 \leq r \leq \infty]$ have surjective homomorphisms onto the reals, where $\text{Diff}^r_+(S^1)$ is the group of orientation-preserving diffeomorphisms of $S^1$ of class $C^r$, with the discrete topology.

REMARKS. The area of $P$ is the hyperbolic deficiency of the sum of the angles of $P$, that is, $\text{Area}(P) = (k-2)\pi - \sum_k \alpha_k$. Note how this information is retained in $F_p$. When the angles $\alpha_k$ all divide $\pi$, $F_p$ may also be obtained from $T_1(H^2) = \text{SL}(2)$ modulo a discrete subgroup, with surgery performed along transverse curves. A similar construction could have been based on surfaces having a number of isolated corners, with metrics of constant negative curvature everywhere else.

These results contrast sharply with the $C^0$ case, for Mather has shown that $B\Gamma_1^0$ is a $K(Z_2, 1)$ $[5]$. It follows that the foliations $F_p$ on $S^3$ bound $C^0$, but not $C^2$, $\Gamma_1$-structures on $D^4$.

In the analytic case, Haefliger has shown that $B\Gamma_1^\omega$ is a $K(\pi_1, 1)$ for some group $\pi_1$ which is uncountable and has a subgroup of index 2 which equals its commutator subgroup $[4]$.

Gel'fand and Fuks $[1]$ give the continuous Eilenberg-Mac Lane cohomology $H^*_c(\text{Diff}^\omega(S^1); R)$ [do they mean $\text{Diff}^\omega_c(S^1)$?] which in dimension two has two generators. The Euler class, and the Godbillon-Vey invariant integrated over the fibres of the natural $S^1$ bundle give two linearly independent classes in $H^2_c(\text{Diff}^\omega_c(S^1); R)$ which map to linearly independent classes in the ordinary Eilenberg-Mac Lane cohomology, but to zero in $H^2_c(\text{Diff}^\omega(S^1); R)$. Perhaps, using the work of Gel'fand and Fuks, it will be possible to compute the continuous Eilenberg-Mac Lane cohomology of $\Gamma_{F_p}^\omega$; that is, the ring of cohomology invariants depending continuously on foliations.
References


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