FILTERED AND ASSOCIATED GRADED RINGS

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1. Introduction. The object of this note is to present a condition which guarantees that a filtered ring $A$ is isomorphic (in the category of filtered rings) to its associated graded ring $\text{gr} A$. The result is that a separated, complete, nonnegatively filtered ring $A$ over a field $k$ of characteristic 0 is isomorphic to $\text{gr} A$ if and only if $\dim k H^2(\text{gr} A, \text{gr} A) = \dim k H^2(A, A)$ where the $\dim k H^2(\text{gr} A, \text{gr} A)$ is finite. The tool is algebraic deformation theory. Rim has observed that an application of the main theorem yields a condition for a plane algebroid curve over an algebraically closed field of characteristic 0 to be of the form $u^m = v^n$ — a result obtained by Zariski [5] by a different approach.

2. Since $A$ is a deformation of $\text{gr} A$ (Gerstenhaber [1]), there exists a one-parameter family of deformations $A_t = \text{gr} A[[t]]$ with multiplication defined by $f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots$. It is known that the deformation from $\text{gr} A$ to $A$ given by $A_t$ is a "pop deformation", i.e., for $t \neq 0$, $A_t$ is isomorphic as a filtered ring to $A[[t]]$ (Gerstenhaber [2]).

Let $\delta_t$ denote the Hochschild coboundary operator of the algebra $A$, i.e., computed relative to the multiplication $f_t$. For example, for $\varphi \in C^1(A, A)$, the group of 1-cochains of $A$, one has

$$\delta_t \varphi(a, b) = f_t(a, \varphi b) - \varphi(f_t(a, b)) + f_t(\varphi a, b).$$

If there exists $\eta_t \in C^1(A, A)$ such that $z_t = \delta_t \eta_t$, then $z_t \in B^2(A, A)$.

$z_0 \in Z^2(\text{gr} A, \text{gr} A)$ is extendible if there exists $z_t \in Z^2(A, A)$ such that

$$z_t = z_0 + tz_1 + t^2z_2 + t^3z_3 + \cdots.$$

Note that every $b_0 \in B^2(\text{gr} A, \text{gr} A)$ is extendible since $b_0 = \delta \eta_0$ implies that $b_t = \delta \eta_0 = b_0 + tb_1 + t^2b_2 + \cdots$ is an extension of $b_0$ where $\eta_0$ is extended linearly over $k((t))$. An extendible class of $H^2(\text{gr} A, \text{gr} A)$ is a $[z_0]$ for which there is a representative $z_0$ which is extendible. $z_0 \in Z^2(\text{gr} A, \text{gr} A)$ is a jump cocycle if there exists an extension $z_t$ of $z_0$ such that $z_t \in B^2(A, A)$. Each $b_0 = \delta \eta_0 \in B^2(\text{gr} A, \text{gr} A)$ is a jump cocycle since $b_t = \delta \eta_0$ is an extension of $b_0$ and $b_t \in B^2(A, A)$. A jump class of $H^2(\text{gr} A, \text{gr} A)$ is a $[z_0]$ for which there exists a representative $z_0$ which is a jump cocycle.

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1 The results announced here are contained in the author's Ph.D. thesis, written under the guidance of Murray Gerstenhaber at the University of Pennsylvania.

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The following theorem is the algebraic analogue of results obtained by Griffiths [3] for normed complexes and for fibered complex-analytic varieties. We assume the vector space dimension, \( \dim_k H^2(\text{gr } A, \text{gr } A) \), is finite.

**Theorem 1.**

\[
\dim_{k((t))} H^2(A_t, A_t) = \dim_k \frac{\text{Extendible classes of } H^2(\text{gr } A, \text{gr } A)}{\text{Jump classes of } H^2(\text{gr } A, \text{gr } A)} = \dim_k \text{E/J}.
\]

**Proof.** To prove that \( \dim_{k((t))} H^2(A_t, A_t) \leq \dim_k E/J \) one shows that a basis \([z_i], i = 1, \ldots, m, \) of \( H^2(A_t, A_t) \) over \( k((t)) \) can be chosen so that \( z_1 = z_0 + tz_1 + t^2z_2 + \cdots, [z_0] \) are linearly independent over \( k \) and \([z_0]\), the coset of \([z_0]\) in \( E/J \), are linearly independent over \( k \). The map \([z_1] \rightarrow [z_0]\) then establishes this inequality. The map: Extendible classes \( \rightarrow H^2(A_t, A_t) \) defined by \([z_0]\) \( \rightarrow [z_1]\) has kernel equal to the jump classes. An elementary argument shows that this map \( E/J \rightarrow H^2(A_t, A_t) \) preserves linear independence. Thus \( \dim_k E/J \leq \dim_{k((t))} H^2(A_t, A_t) \).

The multiplication of \( A_t \) has been defined as \( f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots \).

**Proposition 1.** \( F_t \) is extendible.

**Proof.** Define \( F_t(a, b) = F_1(a, b) + 2tF_2(a, b) + 3t^2F_3(a, b) + \cdots \). \( F_t \) is an extension of \( F_1 \) since \( f_t(a, f_t(b, c)) - f_t(f_t(a, b), c) = 0 \) holds and the formal derivative of this is

\[
f_t(a, f_t(b, c)) + f_t(a, f_t(b, c)) - f_t(f_t(a, b), c) - f_t(F_t(a, b), c) = 0
\]

which is precisely the condition for \( F_t \) to be a \( \delta_t \)-cocycle.

It is important, as Rim observes, that \( F_t \), the derivative of the multiplication \( f_t \), not only is a cocycle of the deformed algebra but is actually intrinsic to the deformed algebra and represents a cohomology class which would not be altered if \( f_t \) were replaced by an equivalent multiplication \( g_t \). This is proved by the following observations. If \( f_t \) and \( g_t \) are equivalent multiplications, then

\[
f_t(a, b) = \psi_t^{-1}(g_t(\psi_t a, \psi_t b))
\]

where \( \psi_t \) is a linear automorphism. The formal derivative of \( \psi_t f_t(a, b) = g_t(\psi_t a, \psi_t b) \) is

\[
\psi_t'(f_t(a, b)) + \psi_t f_t(a, b) = G_t(\psi_t a, \psi_t b) + g_t(\psi_t' a, \psi_t b) + g_t(\psi_t a, \psi_t'b)
\]

where \( G_t \) is the derivative of \( g_t \). From (1) and (2) it follows that

\[
\psi_t^{-1} \psi_t'(f_t(a, b)) + F_t(a, b) = \psi_t^{-1} G_t(\psi_t a, \psi_t b) + f_t(\psi_t^{-1} \psi_t' a, b) + f_t(a, \psi_t^{-1} \psi_t' b).
\]
Therefore \( F_t(a, b) = \psi_t^{-1}G_t(\psi_t a, \psi_t b) + \delta_t \psi_t^{-1}\psi_t(a, b) \) where \( \delta_t \) is defined with respect to \( \psi_t \) multiplication and the cohomology class in \( H^2(A_t, A_t) \) determined by \( F_t \) is not altered by a change of basis.

**PROPOSITION 2**. \( F_t \) is a jump cocycle.

**PROOF**. Let \( \Phi_t \) be an algebra isomorphism of \( A_t \) onto \( A_t \) where \( t \neq 0 \). Then \( \Phi_t(f(a, b)) = f_1(\Phi_t a, \Phi_t b) \) and the derivative of both sides of this expression is

\[
\Phi_t'(f_t(a, b)) + \Phi_t'(F_t(a, b)) = f_1(\Phi_t a, \Phi_t b) + f_1(\Phi_t a, \Phi_t b)
\]

where \( \Phi_t' \) is the formal derivative of \( \Phi_t \). Rewriting this expression yields

\[
F_t(a, b) = f_t(\Phi_t^{-1}\Phi_t(a, b) - \Phi_t^{-1}\Phi_t(f_t(a, b)) + f_t(a, \Phi_t^{-1}\Phi_t b)
\]

and thus \( F_t = \delta_t \Phi_t^{-1}\Phi_t' \).

**THEOREM 2**. A separated, complete filtered ring \( A \) over a field \( k \) of characteristic 0 is isomorphic to \( \text{gr} A \) if and only if \( \dim_k H^2(\text{gr} A, \text{gr} A) = \dim_k H^2(A, A) \) where the vector space \( \dim_k H^2(\text{gr} A, \text{gr} A) \) is finite.

**PROOF**. By [2], \( A[[t]] \) is isomorphic to \( A_t \) for \( t \neq 0 \). The

\[
\dim_k H^2(A[[t]], A[[t]]) = \dim_k H^2(A, A).
\]

It is therefore sufficient to prove that, for \( t \neq 0 \), \( A_t \) is isomorphic to \( \text{gr} A[[t]] \) with multiplication \( f_0 \) if \( \dim_k H^2(\text{gr} A, \text{gr} A) = \dim_k H^2(A_t, A_t) \). By Theorem 2,

\[
\dim_k H^2(A_t, A_t) = \dim_k \frac{\text{Extendible classes of } H^2(\text{gr} A, \text{gr} A)}{\text{Jump classes of } H^2(\text{gr} A, \text{gr} A)}
\]

\[
\leq \dim_k H^2(\text{gr} A, \text{gr} A).
\]

Therefore the \( \dim_k H^2(A_t, A_t) = \dim_k H^2(\text{gr} A, \text{gr} A) \) implies that the jump classes of \( H^2(\text{gr} A, \text{gr} A) = \{ \text{coboundaries} \} \). But \( F_1 \) is a jump cocycle. Thus \( F_1 = \delta \rho_1 \) and \( F_t(a) = a - t\rho_1(a) \) is an isomorphism of \( A_t \) to \( \text{gr} A [[t]] \) with multiplication \( ab + t^2 F_2(a, b) + t^3 F_3(a, b) + \cdots \). Provided \( k \) has characteristic 0 the above argument can be repeated for \( F_2 \) and, in general, for \( F_n \) to show that \( F_n \) is a jump cocycle with the derivative of the appropriate multiplication taken as the extension of \( F_{n-1} \). By the assumption on dimension, \( F_n = \delta \rho_n \). Therefore \( A_t \) is isomorphic to \( \text{gr} A[[t]] \) with multiplication \( f_0 \).

3. Let \( k \) be an algebraically closed field of characteristic 0 and let \( f(x, y) \) be an irreducible power series with coefficients in \( k \). Let \( C \) be the plane curve defined by \( f = 0 \), \( A \) be the local ring of \( C \) and \( m_A \) be the maximal ideal of \( A \).
The Weierstrass Preparation Theorem and Puiseux's Theorem together imply that \( A \subset k[[t]] \). Thus we can define a filtration on \( A \) so that \( F_0A = A \supset F_1A = t \cap A \supset F_2A = t^2 \cap A \supset \cdots \) and form the associated graded ring \( \text{gr} \, A \). We may assume \( \text{gr}_1 A = F_1A/F_2A = 0 \) since otherwise \( t \in A \) implies that \( A = k[[t]] \) and the curve \( C \) would be non-singular.

\( \text{gr} \, A = k[[t^{v_1}, t^{v_2}, \ldots t^{v_r}]] \) with \( v_1 < v_2 < \cdots < v_r \) by definition of the filtration on \( A \). Since \( A \) is the local ring of a plane algebroid curve, \( \text{gr} \, A \) is generated by at most two elements.

The main result of §1 states that \( A \) is isomorphic to \( \text{gr} \, A \) if and only if \( \dim_k H^2(\text{gr} \, A, \text{gr} \, A) = \dim_k H^2(A, A) \). Thus a suitable basis \( \{u, v\} \) of \( m_A \) can be chosen so that the curve \( C \) is of the form \( u^m = v^n \) provided that \( \dim_k H^2(\text{gr} \, A, \text{gr} \, A) = \dim_k H^2(A, A) \). Rim has observed that these results give an alternate form to a result of Zariski [5] which states that \( l(T) = L \) if and only if for a suitable basis \( \{x, y\} \) of \( m_A \) the equation of the curve \( C \) is of the form \( y^n = x^m \) where \((n, m) = 1\) by the irreducibility of the curve \( C \), \( l(T) \) is the length of the \( A \)-module \( T \) (\( T \) is the torsion submodule of the module of Kähler differentials of \( A \)) and \( L \) is the length of the conductor of \( A \) in the integral closure \( \bar{A} \) of \( A \).

**REFERENCES**


