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THE DIFFERENTIAL CLOSURE OF A DIFFERENTIAL FIELD

BY GERALD E. SACKS[1]

Good afternoon ladies and gentlemen. The subject of mathematical logic splits fourfold into: recursive functions, the heart of the subject; proof theory which includes the best theorem in the subject; sets and classes, whose romantic appeal far outweigh their mathematical substance; and model theory, whose value is its applicability to, and roots in, algebra. This afternoon I hope to sketch some theorems about differential fields first derived by model theoretic methods. In particular, I will indicate why every differential field $\mathscr{A}$ of characteristic 0 has a unique prime differentially closed extension called the differential closure of $\mathscr{A}$. Model theory has proved useful in the study of differential fields because the notion of differential closure is surprisingly more complex than the analogous notions of algebraic closure, real closure, or Henselization. The virtue of model theory is its ability to organize succinctly the sort of tiresome algebraic details associated with elimination theory.

The first concepts of model theory are structure and theory. Typic structures are groups, rings and fields. A theory is a set of sentences. A sentence is about the elements of some structure[2]. The language of fields includes plus (+), times (·), equals (=) and variables that stand for elements of fields. A typic sentence in the language of fields says: every polynomial of degree 7 has a root. A typic theory is the theory of algebraically closed fields of characteristic 0 (ACF_0). A structure $\mathscr{A}$ is said to be a model of a theory T if every sentence of T is true in $\mathscr{A}$. Thus the models of ACF_0 are what men call algebraically closed fields of characteristic 0.

Pure model theory, at first thought, appears to be too general to have any mathematical substance. But that hasty thought is given the lie by several theorems, one of which is due to Vaught [1]: Let T be any coun-

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[1] What follows is a verbatim rendering of an invited address given by the author to the 688th meeting of the American Mathematical Society in Cambridge, Massachusetts on October 30, 1971, verbatim save for the deletion of some improprieties that seemed ornamental on first hearing but proved meretricious in cold print. The author wishes to thank Doctor Lenore Blum for teaching him the essentials of differential fields. Preparation of this paper was partially supported by NSF Grant GP-29079; received by the editors January 13, 1972.

[2] I speak here of first order sentences. Second order sentences are about subsets of structures and belong more to analysis than to algebra. True, ideals are subsets rather than elements, but they tend to be finitely generated and consequently first order in nature.

table complete theory; then the number of countable models of T (up to isomorphism) cannot be 2. (For each positive integer $n \neq 2$ there is a T with just n countable models.) Another is due to Morley [2]. His result is not our concern today, but the machinery he developed for the sake of that result was appropriated by Blum [3] to prove the existence of the differential closure.

Before Blum's thesis (1968), all applications of model theory to algebra were corollaries of what logicians call the compactness theorem. The principal figure in the interaction of model theory and algebra is Abraham Robinson; so let me sketch an instance of the compactness theorem I will call Robinson's principle, formulated by him almost twenty years ago. Since then it has influenced everyone who has applied model theory to fields. In its simplest form it says: Let F be a sentence true in every field of characteristic 0; then there exists an integer p_0 such that F is true in every field of characteristic $p \geqq p_0$.

A typic use of Robinson's principle occurs in the proof of the asymptotic form of Artin's conjecture for Q_p, the p-adic numbers, due to Ax and Kochen. Lang showed that Artin's conjecture was true in R_p, the field of formal power series over the integers mod p, for all p. By the full power of Robinson's principle, there exists a structure R_0, a field of characteristic 0, with the property that every sentence true in R_p for all but finitely many p is also true in R_0. Similarly there exists a structure Q_0, a valued field of characteristic 0, that bears the same relation to the Q_p's that R_0 bears to the R_p's. There is nothing unique about R_0 and Q_0, so it is possible to take some simple precautions in the course of their construction that lead to a proof of their isomorphism as valued fields. Then Artin's conjecture holds in R_0 by Lang, hence in Q_0, and so each instance of Artin's conjecture, since it is expressible by a single sentence, holds in Q_p for all but finitely many p. All the essential algebraic facts needed to prove the isomorphism of R_0 and Q_0 can be found in Kaplansky [4]. The most succinct, yet complete, account of the foregoing can be found in Robinson [5]. For generalizations to value groups other than the integers, see Ershov [6].

The theory of differential fields of characteristic 0 (DF_0) is the theory of fields of characteristic 0 augmented by

$$D(x + y) = Dx + Dy;$$

$$D(xy) = xDy + yDx.$$

Robinson set himself the task of defining the theory of differentially closed fields (DCF_0), a theory that had eluded differential algebraists. After reviewing the notions of algebraic closure, real closure and Henselization, he concluded that DCF_0, if it existed, had to have the following three properties:

1. Every differentially closed field is a differential field.

2. Every differential field of characteristic 0 can be extended to some differentially closed field.

3. Let $\mathscr{A}$ be a differential field of characteristic 0, and let S be a finite system of differential equations and inequations in several variables with coefficients in $\mathscr{A}$. Suppose $\mathscr{B}_0$ and $\mathscr{B}_1$ are differentially closed extensions of $\mathscr{A}$. Then S has a solution in $\mathscr{B}_0$ if and only if S has a solution in $\mathscr{B}_1$. (A differential equation in the variables x_i $(1 \leqq i \leqq n)$ is a polynomial in the variables $D^j x_i$ $(1 \leqq i \leqq n, 0 \leqq j \leqq m_i)$. $D^0 x = x$ and $D^{j+1}x = D(D^j x)$. The order of a differential equation is the maximum of $\{m_i | 1 \leqq i \leqq n\}$.)

Properties 1 and 2 are, I am sure you will agree, ineluctable. Property 3 is inspired by Hilbert's nullstellensatz, Sturm's algorithm and Hensel's lemma. Robinson showed on pure model theoretic grounds that at most one theory satisfies 1, 2 and 3. (His result in general terms is: A theory has at most one model completion.) He then showed, with the help of some algebraic facts from Seidenberg [7], that there exists at least one theory satisfying 1, 2 and 3. Finally he christened the unique theory meeting the requirements of 1, 2 and 3 DCF_0.

The one flaw in Robinson's creation of DCF_0 arose from the generality of his method. He gave no suitable axioms for DCF_0; he merely proved they existed. The missing axioms were located by Blum [3]. They consist of DF_0 augmented by:

(a) Every nonconstant polynomial in one variable has a solution.

(b) If $f(x)$ and $g(x)$ are differential equations such that the order of $f(x)$ is greater than the order of $g(x)$, then $f(x)$ has a solution not a solution of $g(x)$.

An interesting feature of Blum's axioms is that they do not mention differential equations in more than one variable, but they nonetheless imply property 3 (above) concerning equations in several variables. The reduction of several variables to one is common in algebra. For example a field is algebraically closed if every nonconstant polynomial in one variable has a solution, but every algebraically closed field satisfies Hilbert's nullstellensatz which concerns polynomials in several variables. The reduction of several variables to one is a general phenomenon discovered by Blum, and applies to all model completions of universal theories (cf. [8, p. 94]).

Suppose $\mathscr{B}$ is a differentially closed extension of $\mathscr{A}$, a differential field of characteristic 0. $\mathscr{B}$ is said to be prime over $\mathscr{A}$ if for each differentially closed $\mathscr{C}$ extending $\mathscr{A}$, there is a monomorphism (preserving the field operations and the derivative) $m:\mathscr{B} \to \mathscr{C}$ extending the identity map on $\mathscr{A}$; i.e. a prime differentially closed extension is contained in every differentially closed extension. Blum [3] showed that every differential field has a prime differentially closed extension and conjectured it was unique. Her result followed from a general result of Morley [2]: If T is an ω-stable theory and $\mathscr{A}$ is a substructure of a model of T, then there exists a prime model extension of $\mathscr{A}$. She had only to verify that DCF_0 was ω-stable, (= totally transcendental) and that amounted to noting that a countable differential field of characteristic 0 had only countably many simple extensions up to isomorphism. (Two simple extensions $\mathscr{A}(b)$ and $\mathscr{A}(c)$, are isomorphic if the identity map on $\mathscr{A}$ can be extended to an isomorphism of $\mathscr{A}(b)$ and $\mathscr{A}(c)$ that takes b to c.)

An archetypic instance of Morley's general approach is the proof of: Let P be a finite system of differential equations and inequations in one variable over $\mathscr{A}$ with a solution in some extension of $\mathscr{A}$; then there exists a finite system $Q \supset P$ such that Q has a solution in some extension of $\mathscr{A}$ and such that all such solutions are isomorphic over $\mathscr{A}$. The proof is general enough to apply to all ω-stable theories. By a direct limit argument it is safe to assume $\mathscr{A}$ is countable. Let $S\mathscr{A}$ be the set of all isomorphism classes of simple extensions of $\mathscr{A}$. For each finite system R over $\mathscr{A}$ let O_R be the set of all simple extensions $\mathscr{A}(b)$ such that b is a solution of R. Recruit the O_R's to serve as a base for the topology of $S\mathscr{A}$. Then $S\mathscr{A}$ is compact, Hausdorff and totally disconnected, hence a Stone space. Since $S\mathscr{A}$ is a countable Stone space, its isolated points are dense. O_P is nonempty, so there exists an $O_Q \subset O_P$ such that O_Q has only one member.

Morley's machinery is based on a certain contravariant functor S that associates a Stone space $S\mathscr{A}$ with each substructure $\mathscr{A}$. The most important property of S is its tendency to preserve limits. A detailed account of S (and of what follows) is given in [**8**].

Recently Shelah [9] has shown that Morley's prime model extension is unique for all ω-stable theories. His proof is an intricate induction on the Morley rank of a simple extension, a generalization of the notion of the order of a differential equation from the finite integers to the countable ordinals. (The importance of ω-stable theories resides in the fact that every simple extension of a model of an ω-stable theory has a Morley rank, which makes it possible to prove theorems about ω-stable theories by induction on rank.) It follows that any two prime differentially closed extensions of a differential field $\mathscr{A}$ of characteristic 0 are isomorphic over $\mathscr{A}$. Let $\bar{\mathscr{A}}$ denote the unique prime differentially closed extension of $\mathscr{A}$. It seems just to call $\bar{\mathscr{A}}$ the differential closure of $\mathscr{A}$.

$\bar{\mathscr{A}}$ is minimal over $\mathscr{A}$ if there is no differentially closed $\mathscr{B} \subset \bar{\mathscr{A}}$ such that $\mathscr{A} \subset \mathscr{B}$, $\mathscr{A} \neq \mathscr{B}$ and $\mathscr{B} \neq \bar{\mathscr{A}}$. It is not known if $\bar{\mathscr{A}}$ is minimal over

$\mathscr{A}$ for any $\mathscr{A}$ of interest. The algebraic closure, real closure and Henselization each possess the appropriate minimality property, so it seems hard to believe the differential closure does not. Let $\mathscr{Q}$ be the rationals. Then $\bar{\mathscr{Q}}$ is the least differentially closed field of characteristic 0. I conjecture $\bar{\mathscr{Q}}$ is minimal over $\mathscr{Q}$. $\bar{\mathscr{Q}}$ is a total mystery save for a result of L. Harrington [**10**] to the effect that $\bar{\mathscr{Q}}$ is computable. (A countable differential field $\mathscr{A}$ is computable if there exists a one-to-one correspondence between $\mathscr{A}$ and the natural numbers that transforms the addition, multiplication and derivative of $\mathscr{A}$ into computable functions.) Harrington's proof combines a Henkin style construction with the finite basis theorem for radical differential ideals. His construction is far from obvious because no algorithm is known for settling the following kind of question: Let P be a finite system of differential equations and inequations in several variables over a differential field $\mathscr{A}$ of characteristic 0; are all solutions of P in all extensions of $\mathscr{A}$ isomorphic over $\mathscr{A}$? Curiously enough there is an algorithm [**7**] for deciding if P has a solution in any extension of $\mathscr{A}$. The algorithm amounts to evaluating several differential equations at the coefficients of P and observing in each case if a nonzero value is obtained.

The Shelah uniqueness proof also supplies many nontrivial automorphisms of $\bar{\mathscr{A}}$. Suppose $b, c \in \bar{\mathscr{A}}$ are conjugate over $\mathscr{A}$; i.e. b and c solve exactly the same differential equations over $\mathscr{A}$. Then the identity map on $\mathscr{A}$ can be extended to an automorphism of $\bar{\mathscr{A}}$ that takes b to c. So there is some hope for further developments in the Galois theory of differential fields. By the way, on general model theoretic grounds, $\bar{\mathscr{A}}$ is minimal over $\mathscr{A}$ if and only if every set of conjugates in $\bar{\mathscr{A}}$ over $\mathscr{A}$ is finite. ($C \subset \bar{\mathscr{A}} - \mathscr{A}$ is a set of conjugates if every finite subset is; $\{c_1, \ldots, c_n\}$ is a finite set of conjugates if every permutation of $\{c_1, \ldots, c_n\}$ solves the same finite systems of differential equations over $\mathscr{A}$ that $\{c_1, \ldots, c_n\}$ does.)

Blum has conjectured: If $\mathscr{A}$ is differentially closed and $b \in \mathscr{A}(b) - \mathscr{A}$ solves some differential equation over $\mathscr{A}$, then $Dx = 0$ has a solution in $\mathscr{A}(b) - \mathscr{A}$. Blum's conjecture trivially implies $\bar{\mathscr{A}}$ is minimal over $\mathscr{A}$.

The theory of differentially closed fields of characteristic p (DCF_p) is currently beset by difficulties. DCF_p bears the same relation to DF_p that DCF_0 bears to DF_0. DF_p includes an axiom that provides a pth root for x whenever $Dx = 0$. In particular, as Carol Wood [**11**] has shown, DCF_p has the three properties imposed on DCF_0 by Robinson; to be precise, DCF_p is the model completion of DF_p. Unfortunately Wood also showed that DCF_p is not ω-stable, and consequently nothing is known about the existence of prime differentially closed extensions of differential fields of characteristic p.

My time is up. Thank you for listening.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139