SURFACES WITH PARALLEL MEAN CURVATURE VECTOR

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Let $M$ be a surface immersed in an Riemannian manifold $\mathbb{R}^m$ of dimension $m$. Let $D$ denote the covariant differentiation of $\mathbb{R}^m$ and $n$ be a normal vector field on $M$. If we denote by $D^*n$ the normal component of $Dn$, then $D^*$ defines a connection in the normal bundle. A normal vector field $n$ is called parallel if $D^*n = 0$.

Let $H$ and $\mathbf{h}$ denote the mean curvature vector and the second fundamental form of $M$ in $\mathbb{E}^m$. It is easy to see that minimal surfaces of a euclidean $m$-space $\mathbb{E}^m$ and minimal surfaces of hyperspheres of $\mathbb{E}^m$ are surfaces of $\mathbb{E}^m$ with parallel mean curvature vector, i.e. $D^*H = 0$. On the other hand, for any analytic function $\varphi \neq 0$ of $z = u + iv$, defined in a neighborhood of the origin in the $(u, v)$-plane, and constants $\alpha, \beta$ with $\alpha > 0$, Hoffman [3], [4] proved that, up to euclidean motions and isothermal coordinate $E(u, v)$, locally there exists one and only one surface in $\mathbb{E}^4$, denoted by $M(\varphi, \alpha, \beta)$, with parallel mean curvature vector $H$ such that $\alpha = |H|$, and $\varphi = \varphi_3, \beta\varphi = \varphi_4$ where $\varphi_3$ and $\varphi_4$ are given in the Lemma of [3]. These surfaces are easy to check that they are contained in either an affine 3-space or an ordinary 3-sphere of $\mathbb{E}^m$ and they are neither minimal surfaces in $\mathbb{E}^m$ nor minimal surfaces of hyperspheres of $\mathbb{E}^m$. Hence, the following problems seem to be interesting.

Problem I. Let $M$ be a surface immersed in a euclidean $m$-space $\mathbb{E}^m$ with parallel mean curvature vector. If $M$ is neither a minimal surface of $\mathbb{E}^m$ nor a minimal surface of a hypersphere of $\mathbb{E}^m$, is $M$ contained either in an affine 3-space or in an ordinary 3-sphere of $\mathbb{E}^m$?

Problem II. If the answer to Problem I is in the affirmative, is $M$ given locally by one of the surfaces $M(\varphi, \alpha, \beta)$?

The main purpose of this paper is to announce the following results. The details will appear elsewhere.

Theorem I. The answer to Problem I is in the affirmative.

Theorem II. The answer to Problem II is in the affirmative.

From theorem I we have the following corollaries.

Corollary 1. Let $M$ be a surface immersed in an $m$-sphere $S^m$ with

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parallel mean curvature vector. If $M$ is neither a minimal surface of $S^m$ nor a minimal surface of a small $(m - 1)$-sphere of $S^m$, then $M$ must be a surface in a (small or great) 3-sphere of $S^m$ with constant mean curvature.

This corollary follows immediately from Theorem I by imbedding $S^m$ as a hypersphere of $E^{m+1}$.

**Corollary 2.** Let $M$ be a compact surface in $E^m$ with parallel mean curvature vector and vanishing Gauss curvature. Then $M$ is a product surface of two plane circles.

This corollary follows immediately from Theorem 1 of [2] and a result of Lawson [5].

**Corollary 3.** Let $M$ be a complete surface in $E^m$ with parallel mean curvature vector. If the Gauss curvature does not change sign, then $M$ is one of the following surfaces:

(i) a minimal surface of $E^m$,
(ii) a minimal surface of a hypersphere of $E^m$,
(iii) a product surface of two plane circles, or
(iv) a product surface of a straight line and a plane circle.

This corollary follows immediately from Theorem 2 of [3] and Theorem I.

Theorem II follows from Theorem I and the construction of $M(\phi, \alpha, \beta)$ and Theorem I is based on the following lemmas.

**Lemma 1.** Let $M$ be a surface immersed in $E^m$ with parallel mean curvature vector and let $R^N$ be the curvature tensor of the normal bundle. If $H \neq 0$, then either $M$ is a minimal surface of a hypersphere of $E^m$ or $M$ has vanishing normal curvature tensor, i.e. $R^N = 0$.

**Lemma 2.** Let $M$ be a surface in $E^m$ with parallel mean curvature vector and vanishing normal curvature tensor. Then $M$ is contained in an affine 4-space of $E^m$.

**References**

4. ———, Surfaces of constant mean curvature in constant curvature manifolds (to appear).