TWO CONSISTENCY RESULTS IN TOPOLOGY
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The present authors proved in 1966 that for any cardinal \( \alpha \), if a Hausdorff space \( X \) is of cardinality \( > 2^{2^{\alpha}} \) then it contains a discrete subspace of cardinality \( > \alpha \). The following result shows that this cannot be improved (even for completely regular spaces).

**Theorem 1.** \( \text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} \& 2^\alpha = \alpha^+ \& 2^{\alpha^+} \text{ is anything reasonable} \& \text{there exists a zero-dimensional } T_2 \text{ space } X \text{ of cardinality } 2^{\alpha^+} = 2^{2^\alpha} \text{ which is hereditary } \alpha \text{-separable}) \).

(A space is hereditary \( \alpha \)-separable iff every subspace in it has a dense subset of cardinality \( \leq \alpha \).)

**Corollary.** It is consistent to assume that \( 2^{\omega_1} \) is big and there exists a zero-dimensional \( T_2 \) space in which the number of all open sets is any cardinal \( \beta \) with \( \omega_1 \leq \beta < 2^{\omega_1} \) and \( \text{cf}(\beta) \neq \omega \).

This relates to a problem raised by J. de Groot.

A space \( X \) is called \( \alpha \)-Lindelöf if every open cover of it can be reduced to a subcover of cardinality \( \leq \alpha \). \( X \) is hereditary \( \alpha \)-Lindelöf iff every subspace of \( X \) is \( \alpha \)-Lindelöf.

**Theorem 2.** Let \( \alpha \) be a given cardinal number. Then \( \text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC} \& \text{GCH} \& \exists \text{ zero-dimensional } T_2 \text{ space } X \text{ so that}
\begin{enumerate}
  \item \( |X| = \alpha^+ \text{ and } X \text{ is hereditary } \alpha \text{-Lindelöf} \);
  \item \( Y \subseteq X \text{ and } |Y| = \alpha^+ \text{ imply that the weight of the subspace } Y \text{ is } \alpha^{++} \);
  \item \( Y \subseteq X \text{ and } |Y| \leq \alpha \text{ imply that } Y \text{ is closed and discrete} \).
\end{enumerate}

**Corollary.** For any given \( \alpha \), it is consistent to assume that there exists a zero-dimensional \( T_2 \) space of weight \( \alpha^{++} \) in which no subspace has the weight \( \alpha^+ \).

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711