THE FUNDAMENTAL FORM OF A FINITE PURELY INSEPARABLE FIELD EXTENSION

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Communicated December 8, 1971

The purpose of this note is to show that to every finite purely inseparable field extension $K/k$ there is associated in a natural way a symmetric cochain $f: K \times \ldots \times K$ ($\gamma$ times) $\to K$ of $K$ with coefficients in itself which we call the "fundamental form" of $K$. Its degree, $\gamma$, depends on certain structural properties of $K$. The fundamental form is a derivation when considered as a function of any one variable, all others being held fixed. (It is almost always a coboundary when viewed as a function of all variables.) If $K$ is a tensor product of two intermediate fields then its fundamental form is a certain symmetric product of the forms of the intermediate fields. A weak converse is known and a strong one conjectured.

References in this note to Nakai are to [4] and [5], those to Keith are to [3].

1. **DEFINITION.** Let $A$ be a commutative $k$-algebra, set $Y^1(A) = \text{End}_k A$ and for every $n > 1$ let $Y^n(A) = Y^n$ be the set of those $n$-cochains $f$ of $A$ with coefficients in itself which are symmetric as functions of all $n$ variables and which have the property that if all but two variables are fixed then $f$ is a two-cocycle when considered as a function of the remaining ones. If $f \in Y^n$, then the $n+1$-cochain $\Delta f$ defined by $\Delta f(a_1, \ldots, a_{n+1}) = a_n f(a_1, \ldots, a_{n-1}, a_{n+1}) - f(a_1, \ldots, a_{n-1}, a_n a_{n+1}) + a_{n+1} f(a_1, \ldots, a_n)$ is in $Y^{n+1}$. This defines the "Nakai operator" $\Delta: Y^n \to Y^{n+1}$. It is easy to verify that for odd $n$, $\Delta$ is identical with the Hochschild coboundary operator $\delta$ restricted to $Y^n$. However, in general, $\Delta^2 \neq 0$ and the $Y^i$ do not form a complex. Those elements of $Y^1$ which are annihilated by $\Delta$ are called "ith order derivations" or simply "i-derivations" and form an $A$-module denoted by $\mathcal{D}^i$. A 1-derivation is an ordinary derivation of $A$ into itself. If $A$ is unital, which we henceforth assume, then we denote by $Y^n_0$ the submodule of $Y^n$ consisting of those cochains in $Y^n$ which vanish when any variable equals 1. We then have $\Delta Y^n_0 \subset Y^{n+1}_0$, and $\mathcal{D}^i \subset Y^i_0$ for all $i$. If $\varphi \in \mathcal{D}^i$, $\psi \in \mathcal{D}^j$ then their composite $\varphi \psi$ is in $\mathcal{D}^{i+j}$ (Nakai). The space $\bigcup_{i=1}^\infty \mathcal{D}^i$ of all "high order derivations" is thus a ring with an increasing filtration. When $A$ is a finite purely inseparable field extension...
K of k Nakai has shown that $\bigcup \mathcal{D}^i = Y_0$, so for some integer $\gamma$ one has $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \ldots \subset \mathcal{D}^\gamma = \mathcal{D}^{\gamma+1} = \ldots = (\text{End}_k K)_0$. In this case we also have that $\phi \in \mathcal{D}^i$ if and only if $\delta{\phi} \in \mathcal{D}^1 \cup \mathcal{D}^{i-1} + \mathcal{D}^2 \cup \mathcal{D}^{i-2} + \cdots + \mathcal{D}^{i-1} \cup \mathcal{D}^1$, a result due to Keith. This gives an alternative inductive definition of “$i$-derivations” which is meaningful for not-necessarily-commutative rings but which possibly differs from Nakai’s for commutative rings other than purely inseparable field extensions. If $\phi \in \mathcal{D}(K/k)$, then it follows from the foregoing that $\Delta^{i-1}\phi \in \mathcal{D}^1 \cup \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^1$ ($i$ times); in particular, $\Delta^{i-1}\phi$ is a derivation as a function of any single variable. To these results we here add the following:

**Theorem.** If $K/k$ is a finite purely inseparable field extension, and if $\gamma$ is the least integer such that $\mathcal{D}^\gamma = (\text{End}_k K)_0$, then $\mathcal{D}(\mathcal{D}^\gamma) = 1$.

It will follow that $\Delta^{i-1}\mathcal{D}^\gamma$ is a one-dimensional $K$-space any generator of which will be called the “fundamental form” of $K/k$.

2. **Proof of the theorem.** An approximate automorphism of order $m$ (“higher derivation” in the terminology of Jacobson [2]) of a not-necessarily-commutative $k$-algebra $A$ is a formal polynomial $\Phi_i = 1 + t\phi_1 + \cdots + t^m\phi_m$ with $\phi_i \in \text{End}_k A$ ($1 = \text{id}^A$) such that

$$\delta{\phi_i} = \phi_i \cup \phi_{i-1} + \phi_2 \cup \phi_{i-2} + \cdots + \phi_{i-1} \cup \phi_1,$$

for all $a, b \in A$. That is, $\Phi_i$ is an automorphism of $A[t]/t^{m+1}$ (cf. [1]). This is equivalent to having

$$\delta{\phi_i} = \phi_i \cup \phi_{i-1} + \phi_2 \cup \phi_{i-2} + \cdots + \phi_{i-1} \cup \phi_1,$$

where $i = 1, \ldots, m$.

It follows that $\phi_i$ is an $i$-derivation under the inductive definition valid for noncommutative rings. Those $i$-derivations which appear in approximate automorphisms will be called “special”. If $\mathcal{A}$ is an extension of the field $k$ and $\mathcal{A} \cong k \otimes_k A$, then $\mathcal{D}(\mathcal{A}) = k \otimes_k \mathcal{D}(A)$, but the analogous assertion is meaningless for special $i$-derivations since the latter in general do not even form an additive group.

Suppose now that $k$ has characteristic $p > 0$ and that $A = k[x]/(x^q - \alpha)$ where $q = p^e$ for some $e > 0$ and $\alpha$ is some element of $k$. Then $A$ has an approximate automorphism $\Phi_0$ of order $q - 1$ which is completely defined by setting $\Phi_0x = x + t$. This implies that

$$\Phi_0x^m = x^m + m \binom{m}{1} x^{m-1}t + m \binom{m}{2} x^{m-2}t^2 + \cdots + t^m,$$

so writing $\Phi_0 = 1 + t\phi_1 + \cdots + t^{q-1}\phi_{q-1}$, it follows that $\phi_i$ is an $i$-derivation sending $x^m$ to $\binom{m}{i}x^{m-i}$ for all $m \geq 0$. It is convenient to denote this $i$-derivation formally by $D^i/i!$, where $D = d/dx$. If we include the case $i = 0$, then $\text{End}_k A$ can be shown to be a free $A$-module having the $D^i/i!$, $i = 0, 1, \cdots, q - 1$, as a basis. Since $(\Phi_0)^p x = x + pt = x$, one
has \((D/i!)^p = 0\) for all \(i > 0\). (Writing \(i = i_0 + i_1p + i_2p^2 + \cdots + i_sp^s\) with \(0 \leq i_0, i_1, \ldots, i_s \leq p - 1\), one has
\[
D/i! = cD^i_0(D^p/p!)^i(D^{p^2}/p^2!)^i \cdots (D^{p^s}/p^s!)^i
\]
where \(c\) is an integer \(\neq 0 \text{ mod } p\). The highest order of any derivation in \(\text{End}_k A\) is therefore \(q - 1\), which is achieved by \(D^q - 1/(q - 1)!\).

Let \(K/k\) be a finite purely inseparable field extension. By Pickert [6] (cf. also Rasala [7]) there is an extension \(\tilde{k}\) (in fact there is a minimal finite one) such that writing \(A = \tilde{k} \otimes_k A\) we have
\[
A \cong \tilde{k}[x_1]/x_1^q \otimes \cdots \otimes \tilde{k}[x_r]/x_r^q
\]
where \(q_1 = p^{e_1}, \ldots, q_r = p^{e_r}\) for some \(e_1, \ldots, e_r > 0\). Denoting the tensor factors of \(A\) by \(A_1, \ldots, A_r\), we have \(\text{End}_k A = \text{End}_k A_1 \otimes \cdots \otimes \text{End}_k A_r\), from which it follows that \(\text{End}_k A\) is generated by \(1\) and the various \(D/i!\). (Therefore \(\bigcup D^i(A) = (\text{End}_k \tilde{A})_0 = \tilde{k} \otimes (\text{End}_k K)_0\), whence \(\bigcup D^i(K) = (\text{End}_k K)_0\). This concise proof of Nakai's result is due to Keith.) The highest order achieved by any derivation in \(\text{End}_k A\) is that of
\[
D^i_{q_1 - 1} \otimes \cdots \otimes D^i_{q_r - 1} / (q_r - 1)!,
\]
whose order is \(\gamma = (q_1 - 1) + \cdots + (q_r - 1)\). Therefore \(D^\gamma = D^\gamma(A) = (\text{End}_k \tilde{A})_0\), and \(D^\gamma / D^\gamma - 1\) is a free \(A\)-module of rank \(1\). It follows that \(D^\gamma = D^\gamma(K) = (\text{End}_k K)_0\) and that \(\dim_k(D^\gamma / D^\gamma - 1) = 1\), as asserted by the theorem.

3. **Symmetric cup products, conjectures.** If \(f\) is a symmetric \(m\)-cochain and \(g\) a symmetric \(n\)-cochain of the \(k\)-algebra \(A\) with coefficients in itself, then we define the symmetric \(m + n\)-cochain \(f \ast g\) by setting
\[
(f \ast g)(a_1, \ldots, a_{m+n}) = (m!n!)^{-1} \sum f(a_{\sigma_1} \cdots a_{\sigma_m})g(a_{\sigma(m+1)} \cdots a_{\sigma(m+n)})
\]
where the sum is taken over all permutations of \(1, \ldots, m+n\). This is meaningful regardless of the characteristic. One can verify that if \(A = k[x]/(x^q - a)\) then the fundamental form of \(A\) can be defined and equals \(D \cup D \cup \cdots \cup D (q - 1\) times), and that if we have a tensor product of such algebras, \(A_1, \ldots, A_r\) with fundamental forms \(f_1, \ldots, f_r\), then the fundamental form of \(A_1 \otimes \cdots \otimes A_r\) is \(f_1 \ast \cdots \ast f_r\). (This is always a coboundary if \(r > 1\).) It follows that if a purely inseparable field extension \(K/k\) is of the form \(K_1 \otimes_k K_2\), and if the fundamental forms of the factors are \(f_1\) and \(f_2\), then that of \(K\) is \(f_1 \ast f_2\). We conjecture conversely that if the fundamental form factors then \(K\) is a tensor product. This has been shown if one puts certain stringent additional conditions on the factors, but the general question is open.

We remark finally that the "exponents" \(e_1, \ldots, e_r\) of \(K/k\) can be determined once \(\dim_k D^i\) is known for \(i = 1, \ldots, \gamma\), and these in turn depend
on the Nakai operator $\Delta$. For $\dim \mathcal{D}^i/\mathcal{D}^{i-1} = \dim \mathcal{D}^i - \dim \mathcal{D}^{i-1}$ is the number of ways of writing $i = i_1 + \cdots + i_r$ with $0 \leq i_l \leq q_l - 1$ ($= p^s - 1$) for $l = 1, \ldots, r$. That is, it is the coefficient of $t^i$ in

$$F(t) = \prod_{l=1}^{r} \frac{1 - t^{q_l}}{1 - t}.$$ 

Thus, knowing $\Delta$ determines $F(t)$, from which the $q_l = p^{s_l}$ can be determined.

References


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