BOUNDARY VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH RAPIDLY INCREASING COEFFICIENTS

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Communicated by Alberto Calderón, February 28, 1972

1. Introduction. The purpose of this note is to present a general existence theorem for variational boundary value problems for quasilinear elliptic operators in divergence form:
\[ A(u) = \sum_{|a| \leq m} (-1)^{|a|} D^a A(x, u, \ldots, \nabla^m u), \]
in the case where the coefficients \( A_{a} \) do not have polynomial growth in \( u \) and its derivatives. The crucial points in the treatment of rapidly (or slowly) increasing \( A_{a} \)'s are that the Banach spaces in which the problems are appropriately formulated are nonreflexive and that the corresponding operators are not bounded nor everywhere defined and do not generally satisfy a global a priori bound. This existence theorem is based upon an extension of the theory of not everywhere defined unbounded pseudo-monotone mappings (Browder [5], [6], Browder-Hess [7]) to the context of complementary systems.

Detailed proofs will appear elsewhere.

The author wishes to thank F. E. Browder for several stimulating conversations.

2. Main results. We will use the following notations. If \( \xi = \{ \xi_a : |x| \leq m \} \in \mathbb{R}^m \) is a \( m \)-jet, then \( \zeta = \{ \xi_a : |x| = m \} \in \mathbb{R}^m \) denotes its top order part and \( \eta = \{ \xi_a : |x| < m \} \in \mathbb{R}^{m-1} \) its lower order part; for \( u \) a derivable function, \( \xi(u) \) denotes \( \{ D^a u : |x| \leq m \} \). The Orlicz space [11] on \( \mathbb{R}^m \) corresponding to an \( N \)-function \( M \) is denoted by \( L_M(\Omega) \) and the closure in \( L_M(\Omega) \) of the simple functions in \( \Omega \) by \( E_M(\Omega) \). The Sobolev space of functions \( u \) such that \( u \) and its distribution derivatives up to order \( m \) lie in \( L_M(\Omega) \) is denoted by \( W^m L_M(\Omega) \); these spaces are identified to subspaces of the product \( \prod_{|a| \leq m} L_M(\Omega) \equiv \prod L_M. \) \( M^{-1} \) denotes the function conjugate [reciprocal] to \( M \) and \( N \ll M \) means that, for each \( \varepsilon > 0 \), \( M(\varepsilon t)/N(t) \to +\infty \) as \( t \to +\infty \).

AMS 1970 subject classifications. Primary 35J60, 35G30, 47H05; Secondary 46B10.

Key words and phrases. Quasilinear elliptic equation, boundary value problem, rapidly increasing coefficients, Orlicz-Sobolev spaces, coercive, noncoercive, pseudo-monotone operator, monotone operator, nonreflexive Banach space, complementary system.

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Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) such that the Sobolev imbedding theorem holds on \( \Omega \). The basic conditions we impose on the coefficients \( A_{a} \) are the following:

(i) Each \( A_{a}(x, \xi) \) is a real-valued function defined on \( \Omega \times \mathbb{R}^n \) which is measurable in \( x \) for fixed \( \xi \) and continuous in \( \xi \) for fixed \( x \).

(ii) There exist \( N \)-functions \( M \) and \( N \) with \( N \ll M, a(x) \in E_{\mathcal{M}}(\Omega) \) and \( b, c \in \mathbb{R}^+ \) such that

\[
|A_{a}(x, \xi)| \leq a(x) + b \sum_{|\beta|=m} M^{-1} M(c_{\xi_{\beta}}) + b \sum_{|\beta|<m} N^{-1} M(c_{\xi_{\beta}}),
\]

if \( |x| = m \),

\[
|A_{a}(x, \xi)| \leq a(x) + b \sum_{|\beta|=m} M^{-1} N(c_{\xi_{\beta}}) + b \sum_{|\beta|<m} M^{-1} N(c_{\xi_{\beta}}),
\]

if \( |x| < m \),

for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \). (This assumption can be weakened using the Sobolev imbedding theorem of [9].)

(iii) For each \( x \in \Omega, \eta \in \mathbb{R}^{n-1} \), \( \xi \) and \( \xi' \) in \( \mathbb{R}^n \) with \( \xi \neq \xi' \),

\[
\sum_{|\alpha|=m} (A_{a}(x, \xi, \eta) - A_{a}(x, \xi', \eta))(\xi_{\alpha} - \xi'_{\alpha}) > 0;
\]

for each \( x \in \Omega, \xi' \) and \( \xi'' \) in \( \mathbb{R}^n \),

\[
\sum_{|\alpha|=m} (A_{a}(x, \xi, \eta) - \xi''_{\alpha})(\xi_{\alpha} - \xi''_{\alpha}) \to +\infty
\]

as \( |\xi| \to +\infty \) in \( \mathbb{R}^n \), uniformly for bounded \( \eta \) in \( \mathbb{R}^{n-1} \).

Let \( Y \) be a \( \sigma(\prod L_{M}, \prod E_{\mathcal{M}}) \) closed subspace of \( W^{m}L_{M}(\Omega) \) on which we impose the condition

(iv) \( Y = \sigma(\prod L_{M}, \prod E_{\mathcal{M}}) \) cl \( Y_{0} \)

where \( Y_{0} = Y \cap W^{m}E_{M}(\Omega) \); here \( M \) is the \( N \)-function involved in condition (ii). Let \( f \in Y_{0}^{\sigma} \). The variational boundary value problem (VBVP) for \( A(u) = f \) with respect to \( Y \) asks for an element \( u \) in \( Y \) such that \( A_{a}(\xi(u)) \in L_{\mathcal{M}}(\Omega) \) for all \( a \) and

\[
a(u, v) = \sum_{|\alpha| \leq m} \int_{\Omega} A_{a}(\xi(u))D^a v \, dx = f(v),
\]

for all \( v \) in \( Y_{0} \).

More generally we consider a one-parameter family of operators

(2) \( A_{t}(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^a A_{a}(x, u, \ldots, \nabla^{m} u, t), \)

where \( t \in [0, 1] \). The coefficients \( A_{a}(x, \xi, t) \) are assumed to satisfy (i), (ii), (iii) for each \( t \); moreover it is assumed that they are continuous in \( (\xi, t) \).
for fixed $x$, that the functions $M, N, a(x)$ and the constants $b, c$ of (ii) can be chosen independently of $t$ and that the convergence in the second part of (iii) is uniform in $t$. Briefly we will say that (i), (ii), (iii) are satisfied uniformly in $t$.

**Theorem 1.** Let $\{A_t : t \in [0, 1]\}$ be a one-parameter family of operators of the form (2) satisfying (i), (ii), (iii) uniformly in $t$. Let $Y$ be a $\sigma(\prod L_M, \prod E_M)$ closed subspace of $W^mL_M(\Omega)$ satisfying (iv). Suppose that $A_1$ is odd and that for each $\sigma(\prod E_M, \prod E_M)$ continuous linear form $f$ on $Y_0$ there exist a constant $K$ and a neighbourhood $N$ of $f$ in $Y_0$ such that for any $g$ in $N$, any $t$ in $[0, 1]$ and any solution $u$ of the VBVP for $A_t(u) = g$ with respect to $Y$, $\|u\| \leq K$. Then, for each $t$ in $[0, 1]$ and each $\sigma(\prod E_M, \prod E_M)$ continuous linear form $f$ on $Y_0$, the VBVP for $A_t(u) = f$ with respect to $Y$ has at least one solution.

Simple examples show that the above VBVP may have no solution if $f$ is arbitrary in $Y_0$. Assumption (iv) is satisfied for instance by $W^mL_M(\Omega)$ or $W^m_0L_M(\Omega) \equiv \sigma(\prod L_M, \prod E_M)\text{cl}(\mathcal{D}(\Omega))$ if $\Omega$ has the segment property. Theorem 1 can be applied in particular to the operator

$$
\sum_{|a| = m} (-1)^{|a|} D^a(p(D^a u)) + \text{lower order terms},
$$

where $p: R \to R$ is any strictly increasing odd continuous function with $p(+ \infty) = + \infty$ and where the lower order terms satisfy some growth condition involving $p$ and a sign condition.

The following result, in which the Dirichlet form $a(u, v)$ is assumed to be coercive, can be derived as in [4] from Theorem 1.

**Theorem 2.** Let $A$ be an operator of the form (1) satisfying (i), (ii), (iii). Let $Y$ be a $\sigma(\prod L_M, \prod E_M)$ closed subspace of $W^mL_M(\Omega)$ satisfying (iv). Suppose that $a(u, u)/\|u\| \to + \infty$ as $\|u\| \to + \infty$ in $Y$ with $A_*(\xi(u)) \in L_\Omega(\Omega)$ for all $\xi$. Then, for each $\sigma(\prod E_M, \prod E_M)$ continuous linear form $f$ on $Y_0$, the VBVP for $A(u) = f$ with respect to $Y$ has at least one solution.

Existence theorems for problems of this type were first obtained by Višik [14], [15] using a priori estimates on $(m + 1)$st derivatives. In the case of coefficients with polynomial growth, monotonicity methods were first applied to these problems by Browder [2]; basic improvements of Browder's original results were given by Leray-Lions [12] who introduced condition (iii) and proved an analogue of Theorem 2, and by Browder [4] who considered noncoercive problems and proved an analogue of Theorem 1. In the case of rapidly increasing coefficients, Donaldson [8] (see also [10]) obtained the simpler version of Theorem 2 corresponding to the situation where the $A_*$'s satisfy a monotonicity con-
dition with respect to all the derivatives of $u$ and where $M$ satisfies the $\Delta_2$ condition. Recently Browder [5] considered equations with top order terms of polynomial growth but lower order terms of rapid growth.

3. Abstract results. The proof of Theorem 1 rests upon general results on nonlinear operators of monotone type in nonreflexive Banach spaces.

**Definition 1.** Let $E$ and $F$ be Banach spaces in duality with respect to a continuous pairing $\langle \cdot, \cdot \rangle$ and let $E_0$ and $F_0$ be subspaces of $E$ and $F$ respectively. Then $(E, E_0; F, F_0)$ is called a complementary system if, by means of $\langle \cdot, \cdot \rangle$, $E_0^\ast$ can be identified to $F$ and $F_0^\ast$ to $E$.

For instance $(\prod L_M, \prod E_M; \prod L_M, \prod E_M)$ is a complementary system, and if we take a $\sigma(\prod L_M, \prod E_M)$ closed subspace $Y$ of $\prod L_M$ and successively define $Y_0 = Y \cap \prod E_M$, $Z = \prod L_M/Y_0$ and $Z_0 = \{f + Y_0 : f \in \prod E_M\}$, then the pairing between $\prod L_M$ and $\prod L_M$ induces a pairing between $Y$ and $Z$ iff $Y$ satisfies (iv), in which case $(Y, Y_0; Z, Z_0)$ is a complementary system; we will refer to it as the complementary system generated by $Y$. An (equivalent) norm $\| \cdot \|_E$ on $E$ will be called admissible if it is lower semi-continuous for $\sigma(E, E_0)$ and satisfies $\langle y, z \rangle \leq \|y\|_E \|z\|_F$ for all $y$ in $E$ and $z$ in $F$, where $\| \cdot \|_F$ is obtained by first restricting $\| \cdot \|_E$ to $E_0$ and then taking the dual norm.

**Definition 2.** Let $(Y, Y_0; Z, Z_0)$ be a complementary system and let $F$ be a dense subspace of $Y_0$. A one-parameter family of mappings $T_t$ of $D(T_t) \subset Y$ into $Z$, $t \in [0, 1]$, is said to define a pseudo-monotone homotopy with respect to $V$ if (a) $V \subset D(T_t)$ for each $t$ and $T$ is finitely continuous from $[0, 1] \times V$ to the $\sigma(Z, V)$ topology of $Z$, (b) for any sequences $u_t$ in $V$ and $t_t$ in $[0, 1]$ such that $u_t \to u \in V$ for $\sigma(Y, Z_0)$, $t_t \to t$, $T_{t_t}(u_t) \to v \in Z$ for $\sigma(Z, V)$ and $\lim sup \langle u_t, T_{t_t}(u_t) \rangle \leq \langle u, v \rangle$, it follows that $u \in D(T_t)$, $T_t(u) = v$ and $\lim \langle u_t, T_{t_t}(u_t) \rangle = \langle u, v \rangle$. In particular, each mapping $T_t$ is pseudo-monotone with respect to $V$ (where the latter is defined in a similar way).

The following two theorems, together with a geometric result of Rao [13], imply Theorem 1. They extend corresponding results by Browder [3], [5].

**Theorem 3.** Let $\{A_t : t \in [0, 1]\}$ be a one-parameter family of operators of the form (2) satisfying (i), (ii), (iii) uniformly in $t$. Let $Y$ be a $\sigma(\prod L_M, \prod E_M)$ closed subspace of $W^mL_M(\Omega)$ satisfying (iv) and let $(Y, Y_0; Z, Z_0)$ be the complementary system generated by $Y$. For each $t$, let $T_t$ be the mapping of $D(T_t) = \{u \in Y : A_\alpha(\xi(u), t) \in L_G(\Omega) \text{ for all } \alpha\}$ into $Z$ defined by $\langle v, T_t(u) \rangle = a_t(u, v)$ for all $v \in Y_0$, where $a_t(u, v)$ is the Dirichlet form associated with $A_t$. Then $\{T_t : t \in [0, 1]\}$ defines a pseudo-monotone homotopy with respect to any dense subspace $V$ of $Y_0$. 
THEOREM 4. Let \((Y, Y_0; Z, Z_0)\) be a complementary system and consider a one-parameter family of mappings \(T_t\) of \(D(T_t) \subset Y\) into \(Z\), \(t \in [0, 1]\), which defines a pseudo-monotone homotopy with respect to any dense subspace \(V\) of a dense subspace \(V\) of \(Y_0\). Suppose that \(T_t\) is odd on \(V\) outside some ball and that for each \(z\) in \(Z_0\) there exists a neighbourhood \(\mathcal{N}\) of \(z\) in \(Z\) such that \(\bigcup \{ T_t^{-1}(\mathcal{N}) : t \in [0, 1] \}\) is bounded in \(Y\). Suppose that \(Y_0\) and \(Z_0\) are separable and that \(Y\) admits an equivalent admissible norm whose restriction to \(Y_0\) is Gâteaux differentiable. Then for each \(t\) in \([0, 1]\), the range of \(T_t\) contains \(Z_0\).

Pseudo-monotonicity was introduced by Brézis [1]; the extension of Brézis' original results to non everywhere defined unbounded mappings in reflexive Banach spaces was carried out by Browder-Hess [7], with applications in Browder [5] to partial differential equations. The concept of pseudo-monotone homotopy is due to Browder [5], [6]. Complementary systems were defined in [9].

ADDED IN PROOF. Theorem 2 also includes the result announced recently by A. Fougères (C. R. Acad. Sci. Paris, February 1972) where \(\bar{M}\) is required to satisfy the \(\Delta_2\) condition.

REFERENCES


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