This note is about the structure of families of open ideals in the ring of power series in two variables. The Hilbert scheme parametrizing them is stratified into locally closed subschemes $Z_T$, whose dimension we calculate. We then discuss some global consequences for families of 0-dimensional schemes on a surface (§1, Corollaries 2, 3). Except in low characteristics, $Z_T$ is locally an affine space (Theorem 2) and is a locally trivial bundle over the complete variety $G_T$ parametrizing graded ideals of type $T$ (Theorem 3).

1. A stratification of the Hilbert scheme. Let $R$ be the ring of power series $k[[x_1, \ldots, x_r]]$ in $r$ variables over an algebraically closed field $k$, with maximal ideal $m$; and let $R_j$ denote the space of forms of degree $j$ in $R$, so that $R = \prod R_j, j = 0, \ldots, \infty$. If $I$ is an ideal in $R$, we let $I_j$ denote the space of forms in $R_j$ which are initial forms of elements of $I$. By the type of $I$ we mean the sequence

$$T(I) = (t_0, t_1, \ldots, t_j, \ldots),$$

where $t_j = \dim_k(R_j/I_j)$.

We will sometimes refer to a type $T$, meaning a specific infinite sequence $(t_0, t_1, \ldots)$. By the length $|T|$ of $T$ we mean $\sum t_j$, if it is finite. The initial degree of $I$ is the smallest $j$ for which $I_j \neq 0$. It depends only on the type of $I$. It is easy to show that if $I$ has finite colength $n$, then $n = |T(I)|$, and $t_j = 0$ if $j \geq n$.

Let $\text{Hilb}^n R$ be the Hilbert scheme parametrizing the family of ideals of colength $n$ in $R$, and $Z_T$ the subscheme parametrizing ideals of a given type $T$ where $|T| = n$. Then we get a stratification (see [7])

$$\text{Hilb}^n R = \bigcup_{|T| = n} Z_T.$$

For the rest of the paper we consider the case $r = 2$, and let $A = k[[x, y]]$. If $I \subset A$ has colength $n$ and initial degree $d$, then

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\[ T(I) = (1, 2, \ldots, d, t_d, \ldots, t_{n-1}, 0, 0, \ldots), \]
\[ 0 \leq t_j \leq j + 1, \sum t_j = n \text{ and } d \geq t_d \geq \ldots \geq t_{n-1}. \]  

The above conditions (2) characterize the sequences that occur as types of an open ideal in \( A \). We assume \( T \) is a sequence satisfying (2), and set

\[ e_j = t_{j-1} - t_j \quad \text{if } j \geq d, \]
\[ = 0 \quad \text{otherwise}. \]

We call \( e_j \) the \textit{jump index} of \( T \). We let \( G_T \) be the subscheme of \( \text{Hilb}^n R \) parametrizing the graded ideals of type \( T \).

\textbf{Theorem 1.}

\[
\dim Z_T = n - \sum e_j(e_j + 1)/2 = n - d - \sum e_j(e_j - 1)/2, \\
\dim G_T = \sum_{j \geq d} (e_j + 1)e_{j+1}.
\]

The proof is by induction on the colength \( n \). We compare ideals \( I \) of type \( T \) having a certain weak normal form with the ideals \( I:x \), which have a type \( T:x \) which depends only on \( T \) and is of length \( < n \).

Let \( T_n = (1, 1, \ldots, 1, 0, 0, \ldots) \), where \(|T_n| = n \) and \( n > 1 \).

\textbf{Corollary 1.}

\[
\dim Z_T = n - 1 = \dim \text{Hilb}^n A \quad \text{if } T = T_n, \\
< n - 1 \quad \text{if } T \neq T_n, \ |T| = n.
\]

\textbf{Proof.} If \( d = 1 \), then \( T = T_n \) by (2) and \( \dim Z_T = n - 1 \) by (4). If \( d > 1 \) then \( \dim Z_T \leq n - d \leq n - 2 \).

Suppose \( X \) is a nonsingular surface, and consider the Chow morphism (see [2]) onto the \( n \)-fold symmetric product \( X^{(n)} \):

\[ w_n : \text{Hilb}^n X \rightarrow X^{(n)}. \]

\( \text{Hilb}^n X \) is a desingularization of \( X^{(n)} \). If \( z \) is a geometric point of \( X^{(n)} \) representing the zero-cycle \( W = \Sigma n_i Q_i \), of degree \( n \) on \( X \), then \( w_n^{-1}(z) \approx \prod \text{Hilb}^n A \) and \( w_n^{-1}(z) \) parametrizes the subschemes of \( X \) having cycle \( W \) (see [2]). This shows

\textbf{Corollary 2.} \( \dim w_n^{-1}(z) = \sum \kappa = 1(n_\kappa - 1) = n - r \).

Let \( \pi \) denote the partition \((n_1, \ldots, n_r)\) of \( n \), and let \( X_\pi \) be the subvariety of \( X^{(n)} \) parametrizing cycles of index \( \pi \), and \( Y_n = w_n^{-1}(X_n) \). Then by Corollary 2,
(5) \[ \dim Y_n = n + r \quad \text{and} \quad \text{cod} Y_n = n - r. \]

Let \( D \) denote the singular locus on \( X^{(n)} \), parametrizing cycles \( \Sigma n_i Q_i \) where some \( n_i \neq 1 \), and let \( B = w_n^{-1}(D) \). \( B \) is the branch locus of the universal subscheme \( Z^n \),

\[ X \times \text{Hilb}^n X \supset Z^n \to \text{Hilb}^n X, \]

over \( \text{Hilb}^n X \) (see [2] or [1]).

**Corollary 3.** \( B \) is irreducible.

**Proof.** By (5), the highest dimensional component of \( B \) is just \( Y_{[2]} \) mod a lower dimensional subvariety, where \( [2] \) denotes the partition \((2, 1, 1, \ldots)\) of \( n \). But \( X_{[2]} \) is irreducible, and the fibers of \( w_n : Y_{[2]} \to X_{[2]} \) are projective lines, so \( Y_{[2]} \) is irreducible. \( Z^n \) is flat over the nonsingular \( \text{Hilb}^n X \), so it is Cohen-Macauley. It must be nonsingular in codimension 1, since any singularity would have to be over \( Y_{[2]} \) and we can reduce to the case \( \text{Hilb}^2 X \) where \( Z^2 \) is nonsingular. Therefore \( Z^n \) is normal, and by purity of branch locus, \( B \) is pure codimension 1. Corollary 3 follows from the irreducibility of \( Y_{[2]} \).

From Corollary 3 one deduces that \( \text{Pic} (\text{Hilb}^n X) \otimes \mathbb{Q} = \mathbb{Q} \oplus \text{Pic} X^{(n)} \otimes \mathbb{Q} \).

2. **The varieties** \( Z_T \) and \( G_T, \ r = 2 \). We assume \( |T| = n \). Except where noted, the results are valid in all characteristics.

**Theorem 2.** \( Z_T \) and \( G_T \) each have a connected cover by Zariski opens in an affine space, hence they are irreducible, rational, and nonsingular. \( G_T \) is also complete.

**Theorem 2'.** If \( \text{char} \ k = 0 \), or \( \text{char} \ k > n \), \( Z_T \) and \( G_T \) each have a connected cover by open sets isomorphic to affine spaces.

The proof uses a normal form for ideals of type \( T \) in \( A \). Theorem 2' arises from a better normal form in those cases.

To each ideal \( I \) we associate the completed graded ideal \( \text{gr}(I) = \prod I_j \), and clearly \( T(I) = T(\text{gr} I) \). This leads to a morphism \( \pi : Z_T \to G_T \), having a section \( s : G_T \to Z_T \) induced by the inclusion of graded ideals of type \( T \) in all ideals of type \( T \).

**Theorem 3.** \( \pi : Z_T \to G_T \) is a locally trivial bundle having fibre an affine space and having a natural "0-section" \( s \). In general, \( Z_T \) is not an algebraic vector bundle over \( G_T \).

These are bundles with group \( \text{Aut}(A) \), where \( A = \text{affine space} \), but whose group cannot in general be reduced to \( \text{Gl}(A) \). In characteristic 0, such a bundle is diffeomorphic (but not algebraically isomorphic) to a vector bundle. See the example below.
The structure of $G_T$ is known only in the simplest cases; for example, if $T$ satisfies $e_j(e_{j+1}) = 0$ for all $j$, then $G_T = \prod_{j > d} P_{e_j}$.

**Example.** If $T = T_n$, then $Z_T$ parametrizes ideals in $A$ of colength $n$, and initial degree 1. Typical such ideals are

$$I_C = (y + c_0x + \ldots + c_{n-2}x^{n-1}, m^n)$$

and

$$I_B = (x + b_0y + \ldots + b_{n-2}y^{n-1}, m^n).$$

Thus $\text{gr}(I_C) = (y + c_0x, m^n)$, and $\text{gr}(I_B) = (x + b_0y, m^n)$.

Every ideal of type $T_n$ has one or both of the above forms; $c_0$ and $b_0$ are coordinates on the two affine pieces of $P_1 = G_T$; and the analysis of $Z_n$ proceeds from an analysis of the transition functions:

$$I_B = I_C \iff \begin{cases} b_0 = g_0(c_0^{-1}) = c_0^{-1} \\ b_j = g_j(c_0^{-1}, c_1, \ldots, c_j) \end{cases}.$$
This conjecture is implied by

**Conjecture 2.** $Z_T$ is in the singular locus of $\text{Hilb}^n A$ if $T \neq T_\nu$.

We have checked Conjecture 2 if $T$ has initial degree 2 and in some other cases—in fact all cases where we have specific parameters for ideals near a given ideal $I$ of colength $n$ in $A$. The analogous conjectures are false in higher dimensions [9].

**BIBLIOGRAPHY**

8. ———, *Families of linear ideals in $k[[x_1, \ldots, x_n]]$: Some locally trivial bundles that are not vector bundles, over $P_1$ and Grass$_n^\nu$* (to appear).
10. ———, *Bundles over $P_1$ with fibre an affine plane* (to appear).

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