This announcement is a continuation of Greene-Wu [1]; we shall present additional theorems relating curvature to function theory on noncompact Kähler manifolds. The first theorem improves Theorem 3 of [1].

**Theorem 1.** Let \( M \) be a complete simply connected Kähler manifold with nonpositive sectional curvature such that, for some \( 0 \in M \),

\[
|\text{sectional curvature} (p)| \leq C(d(0,p))^{-2 - \varepsilon}
\]

for some positive constants \( C \) and \( \varepsilon \), where \( d \) is the distance function associated with the Kähler metric; then \( M \) admits no bounded holomorphic functions.

This theorem is false if \( \varepsilon \leq 0 \). Indeed, on the unit disc, the Kähler metric \( (1 - z^2)^{-n}dzd\overline{z} \) (where \( n \) is any integer \( \geq 3 \)) is complete and its curvature function \( K \) satisfies \( K < 0 \) and \( |K(z)| \leq C(d(0,z))^{-2} \). (0 = origin of \( C \).)

The next theorem and its corollary provide information about the absence of holomorphic \( p \)-forms (\( p \geq 1 \)) when the manifold is positively curved. For compact \( M \), the result was known (Kobayashi-Wu [6]).

**Theorem 2.** Let \( M \) be a complete Kähler manifold of positive scalar curvature; then \( M \) possesses no holomorphic \( n \)-form in \( L^2 \) \((n = \dim M)\). If the eigenvalues \( r_1, \ldots, r_n \) of the Ricci tensor satisfy

\[
r_{i_1} + \ldots + r_{i_p} > 0 \quad \text{for all } i_1 < \ldots < i_p,
\]

then \( M \) admits no holomorphic \( p \)-form in \( L^2 \).

**Corollary.** (A) If \( M \) is a complete Kähler manifold with positive Ricci
curvature, then $M$ admits no holomorphic $p$-form in $L^2$ ($1 \leq p \leq \dim M$). (B) If $M$ is a domain in $C^n$ which admits a complete Kähler metric of positive scalar curvature, then $M$ must have infinite Lebesgue measure.

The next two theorems are concerned with the existence of holomorphic functions.

**Theorem 3.** Let $M$ be a complete Kähler manifold with positive Ricci curvature and nonnegative sectional curvature. Furthermore, let $L$ be a holomorphic line bundle on $M$ with nonnegative curvature. Then $H^p(M, \mathcal{O}(L)) = 0$ for $p \geq 1$.

**Corollary.** (A) Let $M$ be a domain in $C^n$ which admits a complete Kähler metric of positive Ricci curvature and nonnegative sectional curvature; then $M$ is a Stein manifold. (B) Let $M$ be a complete noncompact Kähler manifold with positive sectional curvature; then all the first and second Cousin problems on $M$ are solvable.

Some comments on this theorem follow. First, if $M$ is compact, then this is a special case of Kodaira's vanishing theorem. Second, if the sectional curvature of $M$ is actually positive, then one can even show $H^p(M, \mathcal{O}(T^{(a)} \otimes L)) = 0$ where $L$ is as above and $T^{(a)}$ denotes the $\mu$th symmetric power of the holomorphic tangent bundle of $M$ ($\mu \geq 0$). When $M$ is compact, this statement is a special case of a general theorem due to Griffiths [3, p. 212, Theorem G]. Third, we conjecture that a non-compact complete Kähler manifold $M$ of positive curvature is a Stein manifold.\footnote{We have now proven this conjecture.} We have proven this fact if $M$ in addition possesses a pole, i.e. an $m \in M$ such that $\exp_m : M^m \to M$ is a diffeomorphism. Fourth, the proof of Theorem 3 hinges on a technical lemma which has the following easily stated consequence: every convex function on a Kähler manifold is plurisubharmonic. (A function on a Riemannian manifold is convex if and only if its restriction to each geodesic is a convex function of one variable.) This fact, which is so easy to prove when the Kähler manifold is $C^n$, turns out to be surprisingly subtle in the general case (see the forthcoming paper of Greene-Wu [2]).

For the statement of the next result, we need some notation. Let $A(M)$ be the algebra of holomorphic functions on $M$, let $\Omega$ be the volume form of $M$ and let $\rho$ be the distance (relative to the Kähler metric) from a fixed point $0 \in M$.

**Theorem 4.** Let $M$ be an $n$-dimensional complete simply connected Kähler manifold whose sectional curvature is bounded between $-d^2$ and 0. Then for any $C^2$ plurisubharmonic function $\varphi$ on $M$, the set
\[ \left\{ u \in A(M) : \int_M |u|^2(1 + \rho^2)^{N} \exp\{-(2n - 1)d^2\rho^2 - \varphi\} \Omega < \infty \text{ for some integer } N \right\} \]

is dense in \( A(M) \). If \( M \) satisfies also \(-d^2 \leq \text{sectional curvature} \leq -c^2 < 0\), then the set

\[ \left\{ u \in A(M) : \int_M |u|^2(1 + \rho^2)^{-1} \exp\{-(2n - 1)d^2\rho^2 - \varphi\} \Omega < \infty \right\} \]

is already dense in \( A(M) \).

The first half of this theorem was essentially known to P. A. Griffiths (private communication); in the case \( d = 0 \) and hence \( M = C^\omega \), the theorem is due to Hörmander [4, p. 119].

The next theorem is concerned with boundedness properties of the solutions of \( \partial u = f \). Again, the theorem is due to Hörmander if \( d = 0 \) ([4, p. 107], [5, p. 92]). Let us first explain the notation to come. For a continuous function \( \varphi \) on \( M \) and for an open subset \( D \) of \( M \), we define

\[ L^2(p, \Omega)(D, \varphi) = \left\{ f : f \text{ is a measurable } (p, \varphi) \text{ form on } M \text{ such that} \right\} \]

\[ \int_D |f|^2 e^{-\varphi} < \infty \}

\[ L^2(p, \Omega)(D, \text{loc}) = \left\{ f : f \text{ is a measurable } (p, \varphi) \text{ form on } M \text{ such that} \right\} \]

\[ \int_C |f|^2 \Omega < \infty \text{ for any compact } C \subseteq D \right\} \]

**Theorem 5.** (A) Let \( M \) be a simply connected complete Kähler manifold of dimension \( n \) such that \(-d^2 \leq \text{sectional curvature} \leq 0\). Let \( D \) be a bounded pseudoconvex open set in \( M \), let \( \delta \) be the diameter of \( D \) relative to the Kähler metric and let \( \varphi \) be a plurisubharmonic function in \( D \). For every \( f \in L^2(0, \varphi) (D, \varphi) \), \( q > 0 \), with \( \partial f = 0 \), one can then find \( u \in L^2(0, \varphi - 1)(D, \varphi) \) such that \( \partial u = f \) and

\[ q \int_D |u|^2 e^{-\varphi} \Omega \leq \left( \frac{1}{2} \delta^2 \exp(1 + \frac{1}{2}(2n - 1)d^2\delta^2) \right) \int_D |f|^2 e^{-\varphi} \Omega. \]

(B) Let \( M \) be as in (A). Let \( \varphi \) be any \( C^2 \) plurisubharmonic function on \( M \)
and for a positive integer q, let \( \Phi = \varphi + (2n - 1)qd^2 \rho^2 \). If \( g \in L^2_{(0, q)}(M, \Phi) \) such that \( \delta g = 0 \), then there exists \( u \in L^2_{(0, q-1)}(M, \text{loc}) \) such that \( \delta u = g \) and

\[
2 \int_M |u|^2 e^{-\Phi} (1 + \rho^2)^{-2} \Omega \leq \int_M |g|^2 e^{-\Phi} \Omega.
\]

Finally, we give an improved version of Theorem 4 of [1]. First, we give a new definition of pseudo-Hermitian metrics. On a complex manifold \( M \), \( g \) is called a pseudo-Hermitian metric if and only if (1) \( g \) is a continuous Hermitian bilinear form on \( M \), (2) \( g \) is a \( C^2 \) Hermitian metric outside a proper subvariety \( S \). The emphasis here is that \( g \) is not required to be a continuous tensor on \( M \) and that in specific examples, \( g \) will definitely fail to be differentiable on the singularity set \( S \). By the Ricci curvature or holomorphic sectional curvature of \( g \), we mean that of \( g \) restricted to \( M - S \).

**Theorem 6.** A pseudo-Hermitian metric with nonpositive Ricci curvature on \( C^n \) must satisfy

\[
\limsup_{|z| \to \infty} |z|^2 (\text{Ricci curvature} (z)) > -\infty.
\]

(Here, \( |z|^2 = \sum_{i=1}^n z_i \bar{z}_i \).)

**Corollary.** For \( n > 1 \), every pseudo-Hermitian metric on \( C^n \) must satisfy

\[
\limsup_{|z| \to \infty} |z|^2 (\text{holomorphic sectional curvature} (z)) > -\infty.
\]

It remains to point out that Theorem 6 is false for an exponent > 2. Indeed, the pseudo-Hermitian metric \((1 + |z|^\delta)dz \bar{dz}\) on \( C \) (where \( \delta \) is any positive constant) satisfies

\[
\lim_{|z| \to \infty} |z|^{2+3\delta} (\text{curvature} (z)) = -\infty.
\]

(We take this opportunity to rectify some errors in [1]. (A) Theorem 1(iii) should be amended to read: If sectional curvature \( \leq -c^2 < 0 \), then \( dd^c \rho^2 \geq (4 + 2c \rho \coth c \rho) \omega \), \( dd^c \log (1 + \rho^2) > 0 \) and outside \( \{m: \rho(m) < 1\} \),

\[
dd^c \log (1 + \rho^2) \geq 2(\rho c \coth c - 1)/(1 + \rho^2)
\]

where \( \coth \) denotes the hyperbolic cotangent. (B) The conclusion of Theorem 2(ii) should be

\[
\int_{S_r} |f|^p \omega_r \geq D_f \exp \{(2n - 1)^{1/2} cr\}
\]
for \( r \geq 1 \) and for some \( D_f \) which is independent of \( r \) and is positive if \( f(0) \neq 0 \).)

**BIBLIOGRAPHY**


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