1. Introduction. When Sinai [11], [12] and Bowen [1], [2] studied invariant measures for an Anosov diffeomorphism, or on basic sets for an Axiom A diffeomorphism, they encountered problems reminiscent of statistical mechanics (see [10, Chapter 7]). Sinai [13] has in fact explicitly used the techniques of statistical mechanics to show that an Anosov diffeomorphism does not in general have a smooth invariant measure.

We rewrite here a part of the general theory of statistical mechanics for the case of a compact set $\Omega$ satisfying expansiveness and the specification property of Bowen [1]. Instead of a $Z$ action we consider a $Z^*$ action as is usual in lattice statistical mechanics (where $\Omega = F^{Z^*}$ with $F$ a finite set). This rewriting presents a number of technical problems, but the basic ideas are contained in the papers of Gallavotti, Lanford, Miracle-Sole, Robinson, and Ruelle [5], [7], [8], [9], etc.

2. Notation and assumptions. Given integers $a_1, \ldots, a_v > 0$, let $Z^*(a)$ be the subgroup of $Z^*$ with generators $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, a_v)$. We write also

$$\Lambda(a) = \{ m \in Z^* : 0 \leq m_i < a_i \},$$

$$\Pi(a) = \{ x \in \Omega : Z^*(a)x = \{ x \} \}.$$

If $(\Lambda_a)$ is a directed family of finite subsets of $Z^*$, $\Lambda_a \to \infty$ means $\text{card } \Lambda_a \to \infty$ and $\text{card}(\Lambda_a + F)/\text{card } \Lambda_a \to 1$ for every finite $F \subset Z^*$. In particular $\Lambda(a) \to \infty$ when $a \to \infty$ (i.e. when $a_1, \ldots, a_v \to \infty$).

Let $Z^*$ act by homeomorphisms on the metrizable compact set $\Omega$, and let $d$ be a metric on $\Omega$. $C(\Omega)$ is the Banach space of real continuous functions on $\Omega$ with the sup norm, and $C(\Omega)^*$ the space of real measures on $\Omega$ with the vague topology. The two assumptions below will be made throughout what follows.

2.1. Expansiveness. There exists $\delta^* > 0$ such that

$$(d(mx, my) \leq \delta^* \text{ for all } m \in Z^*) \Rightarrow (x = y).$$

2.2. Specification. Given $\delta > 0$ there exists $p(\delta) > 0$ with the following property. If $(\Lambda_a)$ is a family of subsets of $\Lambda(a)$ such that the sets $\Lambda_i + Z^*(a)$
have mutual (Euclidean) distances $p(\delta)$, and if $(x_i)$ is a family of points of $\Omega$, there exists $x \in \Pi(a)$ such that
\[ d(mx_i, mx) < \delta \]
for all $m \in \Lambda$, all $l$.

If $\Omega$ is a basic set for an Axiom A diffeomorphism ($v = 1$), it is known that expansiveness [14] holds, and that specification [1] holds for some iterate of the diffeomorphism.

3. **Pressure and entropy.** Letting $\delta > 0$, we say that $E \subset \Omega$ is $(\delta, \Lambda)$-separated if
\[ ((x, y) \in E \text{ and } d(mx, my) < \delta \text{ for all } m \in \Lambda) \Rightarrow (x = y). \]

Let $\phi \in C(\Omega)$. Given $\delta > 0$ and a finite $\Lambda \subset \mathbb{Z}^v$, or given $a = (a_1, \ldots, a_n)$ we introduce the "partition functions"
\[ Z(\phi, \delta, \Lambda) = \max_E \sum_{x \in E} \exp \sum_{m \in \Lambda} \phi(mx), \]
where the max is taken over all $(\delta, \Lambda)$ separated sets, or
\[ Z(\phi, a) = \exp \sum_{m \in \Lambda(a)} \phi(mx). \]

3.1. **Theorem.** If $0 < \delta < \delta^*$, the following limits exist:
\[ \lim_{\Lambda \to \infty} \frac{1}{\text{card } \Lambda} \log Z(\phi, \delta, \Lambda) = P(\phi), \]
\[ \lim_{a \to \infty} \frac{1}{\text{card } \Lambda(a)} \log Z(\phi, a) = P(\phi), \]
where $P$ defines a real convex function on $C(\Omega)$ such that
\[ |P(\phi) - P(\psi)| \leq \|\phi - \psi\|; \]
$P$ is called the pressure.

Other definitions of $P$, using open coverings or Borel partitions of $\Omega$, are possible.

Let $\mathcal{A} = (A_j)_{j \in J}$ be a finite Borel partition of $\Omega$, and $\Lambda$ a finite subset of $\mathbb{Z}^v$. We denote by $\mathcal{A}^\Lambda$ the partition of $\Omega$ consisting of the sets $A(k) = \bigcap_{m \in \Lambda} (-m)A_{k(m)}$ indexed by maps $k: \Lambda \to J$. We write
\[ S(\mu, \mathcal{A}) = -\sum_j \mu(A_j) \log \mu(A_j). \]
Let $I$ be the (convex compact) set of $\mathbb{Z}^v$ invariant probability measures on $\Omega$.

3.2. **Theorem.** If $\mathcal{A}$ consists of sets with diameter $\leq \delta^*$ and $\mu \in I$, then
This limit is finite \(\geq 0\), and independent of \(\mathscr{A}\). Furthermore, \(s\) is affine upper semi-continuous on \(I\); \(s\) is called the entropy.

For \(\nu = 1\), this is the usual definition of the measure theoretic entropy. Specification is not used in the proof of Theorem 3.2.

4. Variational principle and equilibrium states. Let \(I\) be the set of \(\mu \in C(\Omega)^*\) such that

\[
P(\phi + \psi) \geq P(\phi) + \mu(\psi) \quad \text{for all } \psi \in C(\Omega).
\]

Those \(\mu\) are called equilibrium states for \(\phi\).

4.1. Theorem. The following variational principle holds:

\[
P(\phi) = \max_{\mu \in I} [s(\mu) + \mu(\phi)].
\]

The maximum is reached precisely for \(\mu \in I_\phi\) (in particular \(I_\phi \subset I\)). The set \(I_\phi\) is not empty; it is a Choquet simplex, and a face of \(I\) [3]. There is a residual subset \(D\) of \(C(\Omega)\) such that \(I_\phi\) consists of a single point \(\mu_\phi\) if \(\phi \in D\). For all \(\mu \in I\),

\[
s(\mu) = \inf_{\phi \in C(\Omega)} [P(\phi) - \mu(\phi)].
\]

If \(\Omega\) is a basic set for an Axiom A diffeomorphism it is known [2] that \(0 \in D\), and (*) for \(\phi = 0\) is related to the fact that the topological entropy is the sup of the measure theoretic entropy [4], [6]. Further results on \(D\) have been obtained for Anosov diffeomorphisms using methods of statistical mechanics [13].

4.2. Theorem. Let \(\mu_{\phi,a}\) be the measure on \(\Omega\) which is carried by \(\Pi(a)\) and gives \(x \in \Pi(a)\) the mass

\[
\mu_{\phi,a}(\{x\}) = Z(\phi, a)^{-1} \exp \sum_{m \in \Lambda(a)} \phi(mx).
\]

If \(\mu\) is a (vague) limit point of the \((\mu_{\phi,a})\) when \(a \to \infty\), then \(\mu \in I_\phi\). In particular, if \(\phi \in D\),

\[
\lim_{a \to \infty} \mu_{\phi,a} = \mu_\phi.
\]

REFERENCES


