In this note we describe the theory of singular integral and multiplier transformations in the setting of the spaces $\Lambda(B, X)$. First we introduce the spaces $\Lambda(B, X)$ combining Theorems A and B below, essentially due to A. P. Calderon, with paragraphs 14 and 34 of [2]. Theorem C and the example that follows it illustrate the fact that $\Lambda(B, X)$ spaces are related to Lipschitz spaces of functions and distributions in $\mathbb{R}^n$. (See also [5].) Then guided by the translation invariance of an important class of singular integrals, described before Theorem D, we define a class of operators which commute with representations of $\mathbb{R}^n$ into a group of (uniformly) bounded linear operators of a Banach space $B$ into itself. The continuity of these singular integral operators is proved in Theorem D. Theorems E and F concerning multipliers are then proved with the assumption that the representations alluded to above are the translations. These results were submitted as a thesis at the University of Chicago. I would like to thank Professor A. P. Calderón for having had the privilege of learning with him these and many other things and to Professor Max Jodeit, Jr. for his help throughout my graduate studies.

The spaces $\Lambda(B, X)$. Let $\{t^P\}_{t>0}$ be a group of transformations of $\mathbb{R}^n$ where $P$ is a real $n \times n$ matrix and

$$\tag{*} \|t^P\| \leq t, \quad 0 < t \leq 1.$$ 

This will ensure the existence of a unique value $s$ for which $s^{-P}x \in S^{n-1}$, $x \neq 0$. Thus setting $\rho(x) = s$ we have that $\rho(t^P x) = t \rho(x)$ and $\rho(x + y) \leq \rho(x) + \rho(y)$. (See [4].) We notice that $(P x, x) \geq (x, x)$ is a necessary and sufficient condition for $(*)$ to hold. Moreover the adjoint matrix $P^*$ also satisfies $(P^* x, x) \geq (x, x)$ and therefore it determines a function $\rho^*(x)$ with similar properties. Now we construct a one-parameter family of dilations $\nu_t$ of a finite Borel measure $\nu$ on $\mathbb{R}^n$ by setting $\nu_t(E) = \nu(t^{-P} E)$ for every $\nu$-measurable set $E$ and $t > 0$. If $d\nu(x) = \phi(x) dx$ where $\phi \in L^1(\mathbb{R}^n)$, then $d\nu_t(x) = t^{-n} \phi(t^{-P} x) dx$.

Let $B$ be a Banach space of tempered distributions on $\mathbb{R}^n$ such that $\mathcal{S}(\mathbb{R}^n) \subset B$ and $B = V^*$ for some complex Banach space $V$. For $y \in \mathbb{R}^n$...
let \( \tau_y \) be a representation of \( \mathbb{R}^n \) into a group of (uniformly) bounded linear operators of \( B \) into itself, i.e. \( \| \tau_y \|_B \leq C \| u \|_B \), such that, for each \( u \in B \) and \( v \in V \), \((\tau_y, u, v)\) is a continuous function of \( y \). We now define \( \int \tau_y d\nu(y) \) to be the element \( w \in B \) such that \((w, v) = \int (\tau_y, u, v) d\nu(y) \) for all \( v \in V \). Analogously we define \( F(y, t) \) acting on \( v \in V \) as \((F(y, t), v) = \int (\tau_{y+2t}, u, v) d\nu_t(x) \).

**THEOREM A.** Let \((\tau_y, u, v)\) be a continuous function of \( y \in \mathbb{R}^n \) for \( u \in B \) and \( v \in V \). Let \( v \) satisfy (i) \( \int |x^M| |d\nu(x)| \leq C_M < \infty \) for all multi-indices \( M \) of nonnegative integers and (ii) \((v_t^t)(x) = \hat{\nu}(t^p x) \neq 0 \) as a function of \( t > 0 \) for any \( x \in \mathbb{R}^n - (0) \). Then there exist functions \( \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \) such that

\[
(u, v) = \int (\tau_y, u, v) \phi(y) dy + \lim_{t \to 0} \int (F(y, t), v) \psi_t(y) dy dt.
\]

In fact these functions may be chosen so that \( \hat{\phi} \in C_0^\infty(\mathbb{R}^n) \) vanishes in a neighbourhood of the origin and \( \hat{\psi} \in C_0^\infty(\mathbb{R}^n) \).

We would like to construct a similar representation for elements in \( \Lambda(B, X) \). In order to do so we introduce some definitions.

A lattice \( X \) of locally integrable functions in \((0,1)\) is a linear class of functions such that there is a norm defined on \( X \) with respect to which it is complete and if \( f \in X \) and \( |g| \leq |f| \), then \( g \in X \) and \( \|g\|_X \leq \|f\|_X \).

Given a positive monotone increasing (in the wide sense) submultiplicative function \( p(t) \) defined on \((0, \infty)\) we say that \( X \) is a \( \gamma \)-lattice if the mappings

\[
\begin{aligned}
f \mapsto \int_0^t f(s) p(t/s) (t/s)^{-\epsilon} \frac{ds}{s} \quad \text{and} \quad f \mapsto \int_1^t f(s) (t/s) (t/s)^{-\epsilon} \frac{ds}{s}
\end{aligned}
\]

are continuous from \( X \) into itself for \( \epsilon > 0 \). We choose to call a \( t' \)-lattice an \( r \)-lattice. We set \( \gamma X = \{ f \in L_{1,\infty}(0,1) : \gamma(t)^{-1} f(t) \in X \} \), \( \| f \|_{\gamma X} = \| \gamma^{-1} f \|_X \).

Given \( B \) and \( X \) we denote by \( X(B) = \{ F : F \) is \( B \)-valued weakly measurable and \( \| F \|_X \leq \|F\|_B \} \). Finally we let \( \Lambda_t(B, X) = \{ u \in B : \int \tau_y d\nu(y) (y)(x) \in X(B) \} \). Normed with \( \| u \|_{\Lambda_t} = \| u \|_B + \| \int \tau_y d\nu(y) (y)(x) \|_X \), \( \Lambda(B, X) \) becomes a Banach space in which \( B \) is continuously embedded.

Let \( A_k \) be the class of \( L_1(\mathbb{R}^n) \) functions \( f(x) \) such that there exists a “polynomial” \( \sum_{|M| \leq k} a_M(x) y^M \) so that

(i) \( a_M(x) \in L_1 \) and

(ii) \( \int |f(x - y) - \sum a_M(x) y^M| \, dx = O(|y|^k) \) for \( y \in \mathbb{R}^n \).

**THEOREM B.** Let \( k \in \mathbb{Z}^+ \) and let \( \mu \) be a Borel measure such that (i) \( \int |x^M| d\mu(x) = 0 \) for \( |M| < k \); (ii) \( \int |x|^k |d\mu(x)| < \infty \). Further let \( X \) be a \( \beta \)-lattice and let \( \gamma(t) \) be such that \( \beta(t) \gamma(t)/t^k \) increases for some \( \epsilon > 0 \) and \( \beta(t) \gamma(t)/t^{k-\delta} \) decreases for some \( k > \delta > 0 \). Then for fixed functions \( \phi, \psi \in A_k \) and
elements \( u \in B, F(t) \in X(B) \) the integrals
\[
\mathcal{I}(u, F) = \int \tau_x \mu \phi(y) \, dy + \int_0^1 \int \tau_y F(t) \psi(t) y(t) \, dy \frac{dt}{t}
\]
converge absolutely in the \( B \)-norm and
\[
\| \mathcal{I}(u, F) \|_{A(B, X)} \leq C(\| u \|_B + \| F \|_{X(B)}),
\]
\( C \) independent of \( u, F \).

**Corollary B.1.** If the hypotheses above hold and \( \gamma(t) \equiv 1 \), and \( \mu = v, \phi, \psi \) are as in Theorem A, then \( \mathcal{I}(u, F) \) maps \( B \oplus X(B) \) onto \( \Lambda(\mathbb{R}^n) \).

Also if now the measures \( \mu \neq v \) satisfy both the hypotheses of Theorems A and B, then \( \Lambda(\mathbb{R}^n) \) and \( A(\mathbb{R}^n) \) coincide algebraically and topologically. This explains the denotation \( \Lambda(B, X) \) for these spaces from now on.

**Corollary B.2.** [Cf. [1, paragraph 6.1.]] Let \( \phi \in L^1(\mathbb{R}^n), \supp \phi \) compact and let \( X \) be an \( r \)-lattice, \( 0 < r < 1 \). Then \( \| u - \int \tau_x \mu \phi(y) \, dy \|_B \in X \) implies \( \| u - \tau_z P u \|_B \in X \) for any \( z \in \mathbb{R}^n \).

We now characterize the spaces \( \Lambda(B, X) \) when \( P = \text{diag}(a_1, \ldots, a_n) \), \( a_i \in \mathbb{Z}^+ \). (For \( P = \text{identity} \) see [3, paragraph 14.1].) Put \( a = (a_1, \ldots, a_n) \). Let
\[
v_z(y) = \sum_{j=0}^k \binom{k}{j} (-1)^j \delta(y - j^p z),
\]
where \( \delta \) is the Dirac measure centered at the origin. Moreover, let
\[
\Lambda_t z u = \int \tau_x \mu \, dv_z(y) = \sum_{j=0}^k \binom{k}{j} (-1)^j \tau_z^{(j)} P_z u.
\]
We then have

**Theorem C.** Let \( P, v, \Lambda_t z \) be as above and let \( X \) be an \( r \)-lattice. Furthermore assume that \( (\tau_x, v) = (u, \tau_z) \), i.e. the \( \tau_x \) are the adjoints of a family \( \tau_z \) acting on \( V \). This assumption will be kept for the remainder of the note. Then for multi-indices \( M \) satisfying \( 0 < r - (a, M) < k \) we have
\[
\Lambda(B, X) = \{ u \in B : (\partial / \partial x)^M \tau_x u \big|_{x=0} \in B \text{ and } \sup_{[x]} \| t^{(a,M)} \Lambda_t z (\partial / \partial x)^M \tau_x u \big|_{x=0} \| B \in X \}.
\]
Moreover
\[
\| u \|_\Lambda \leq \sup \{ \| (\partial / \partial x)^N \tau_x u \big|_{x=0} \| B, \| t^{(a,M)} \Lambda_t z (\partial / \partial x)^M \tau_x u \big|_{x=0} \|_{X(B)} \}
\]
the supremum being taken over \( z \in S^{n-1}, (a, N) < r, (a, M) < r \). For example, let \( B = L^p(\mathbb{R}^n), 1 < p \leq \infty \), put \( \tau_x \mu = u(x \Delta^3 y) \) and \( X = X_{r,p} \cap X_{s,p} \).
where $X_{r,p}$ is the $r$-lattice $t' L^p(0,1; dt/t)$. If $a = (1, \ldots, 1, 2)$, $z = (z_1, \ldots, z_{n-1}, 0) \in S^{n-1}$, and $\tilde{z} = (0, \ldots, 0, 1)$ we obtain $\Lambda(B, X) = \{ u \in L^p(R^n) : (\partial/\partial x)^M u \in L^p(R^n) \}$ for $M = (M_1, \ldots, M_n)$ with $M_i \leq 2$ for $1 \leq i \leq n - 1$ and $M_n \leq 1$ and $\| t^2 \Delta_{r,z}(\partial/\partial x)^M u \|_{X_{r,p}(L^r)} + \| t^2 \Delta_{r,\tilde{z}}(\partial/\partial x)^M u \|_{X_{r,p}(L^r)} < \infty$ for $(a, M) = 2$ and $2 < r, s < k + 2$. This is a "parabolic" Lipschitz space of functions with the last variable distinguished.

**Singular integrals.** Let $k \in \mathcal{S}'(R^n)$ be defined by $k(\phi) = \text{p.v.} \int k(x)\phi(x) \, dx$ for $\phi \in \mathcal{S}(R^n)$, where $k(x) \in L^1_{\text{loc}}(R^n - (0))$ and it satisfies

(i) for $0 < r < R$, $\left| \int_{r < \rho(x) < R} \rho(x)k(x) \, dx \right| \leq C$ and $\int_{r < \rho(x) < 1} k(x) \, dx$ converges as $r \to 0$;

(ii) for $R > 0, \int_{\rho(x) < R} \rho(x)|k(x)| \, dx \leq CR$ and

(iii) $\int_{\rho(x) > 4\rho(y)} \nu(x - y) - k(x) \, dx \leq C$. [See [4] and [6].]

We define for $u \in B$ the singular integral $Ku$ as

$$(Ku, v) = \lim_{\epsilon \to 0} \int_{|y| < 1/\epsilon} (\tau_u \phi(y)) k(y) \, dy, \quad \text{for all } v \in V.$$ 

The next theorem is better understood if we recall the following remark due to Taibleson [7, p. 828]. The Riesz transforms $R_t$ defined by $(R_t u)(x) = \frac{x_t}{|x|} u(x)$ are not (in the notation of [7]) bounded mappings of $\Lambda(\alpha, p, \infty)$ into itself for $p = 1, \infty$.

**Theorem D.** Let $A = U^*$, $B = V^*$, $C = W^*$ be Banach spaces of tempered distributions such that $K : A \to B$ continuously. Further assume that $\mathcal{S}$ is dense in $V \cap W$ and that if $\phi(x)$ is the function of Theorem A then $\tau_\phi \psi(y) : B \to C$ continuously. Then

(i) $\tau_x K = K \tau_x$ and $(\tau_x Ku, v)$ is a bounded function of $z \in R^n$ for $u \in A$ and $v \in V$.

(ii) $K$ is a continuous mapping from $A \cap \Lambda(C, X)$ into $B \cap \Lambda(C, X)$.

The proof uses Theorems A and B.

**Multipliers.** In this section we assume that the $\tau_x$ act on $\mathcal{S}'(R^n)$ by $(\tau_x \phi)(y) = \phi(y - x)$ for all $\phi \in \mathcal{S}(R^n)$. A function $m(x)$ continuous and bounded in $R^n - (0)$ is said to be a *multiplier* of type $(A, B)$, where $\mathcal{S}$ is dense in $A$, if the mapping $M$ defined by $(Mu)(x) = m(x)u(x)$, $u \in \mathcal{S}$, satisfies $\| Mu \|_B \leq C\| u \|_A$. Clearly $M\tau_x = \tau_x M$.

**Theorem E.** Let $A = U^*$, $B = V^* \subset \mathcal{S}'(R^n)$. Let $M : U \to V$ be such that $M^{*}\tau_x = \tau_x M^{*}$, $M^{*} : A \to B$ continuously. If $\tau_x$ maps boundedly $A$ into itself and $B$ into itself, then $M^{*} : \Lambda(A, X) \to \Lambda(B, X)$ continuously.

The following is a particular instance of a more general valid fact.

**Theorem F.** Let $A, B$ be as above and let $C = W^* \subset \mathcal{S}'(R^n)$ be such that $\int \tau_x \phi(y) \, dy : B \to C$, where $\phi(y)$ is as in Theorem A. Let $\mathcal{S}(R^n)$ be dense.
in $A \cap C$ and $V \cap W$. If $m(x)$ is $[n/2] + 1$ continuously differentiable in $R^n - (0)$ and if for $\Omega = \{0 < \varepsilon < \rho^*(x) < \varepsilon^{-1}\}$, where the choice of $\varepsilon$ depends solely on the function $\psi(x)$ of Theorem A, we have that

$$\sum_{|M| < [n/2] + 1} \int_{\Omega} |(\partial/\partial z)^m(\rho^* z)|^2 \, dz \leq C \quad \text{for } t \geq 1,$$

then $m(x)$ is a multiplier of type $(A \cap \Lambda(C, X), B \cap \Lambda(C, X))$.

REFERENCES


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