GROUP ACTIONS ON POINCARE DUALITY SPACES

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Let \( G = \mathbb{Z}_p \) for \( p \) prime and \( K = \mathbb{Z}_p \), or let \( G = S^1 \) and \( K = \mathbb{Q} \), and let \( G \) act on the compact space \( X \). In this paper, we outline two proofs of the following:

**Theorem.** Suppose the compact \( G \)-space \( X \) is a Poincaré duality space over \( K \) of formal dimension \( n \). Then each connected component of the fixed point set is a Poincaré duality space over \( K \), and, if \( G \neq \mathbb{Z}_2 \), has formal dimension congruent to \( n \) mod 2.

This solves affirmatively the conjecture of Su given in [5].

Let \( E_G \to B_G \) be the universal bundle for \( G \) and let \( X_G \) be the balanced product \((X \times E_G)/G\). The basic tools for both proofs are the fibre space \( X \to X_G \to B_G \) and the localization theorem of Borel ([1], [4]). In the case \( X \) is totally nonhomologous to zero in \( X_G \), Bredon has proven the Su conjecture [2]. However, this condition can be replaced by the two lemmas below, and this constitutes our algebraic proof. The second proof involves applying the localization theorem to a Thom space.

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1. **Algebraic proof.** When \( G = S^1 \) or \( \mathbb{Z}_2 \), \( H^*(B_G) = K[t] \) where \( t \) is of degree two in the \( S^1 \) case and of degree one in the \( \mathbb{Z}_2 \) case. If \( G = \mathbb{Z}_p \) for \( p \) odd, then \( H^*(B_G) = K[t, s]/s^2 = 0 \) where \( s \) has degree one and \( t \) degree two. We consider the cohomology spectral sequence of the fibre space \( X \to X_G \to B_G \).

**Lemma 1.** (1) \( E_r \) is generated over \( K[t] \) by \( E_{r+1}^r \) and \( E_r^1 \) for \( G \neq \mathbb{Z}_2 \) or \( S^1 \), and by \( E_r^0 \) for \( G = \mathbb{Z}_2 \) or \( S^1 \).

(2) If \( j \geq r - 1 \), cup product with \( t \) gives an isomorphism of \( E_r^{j,k} \) into \( E_r^{j+2,k} \) for \( G \neq \mathbb{Z}_2 \) and of \( E_r^{j,k} \) into \( E_r^{j+1,k} \) for \( G = \mathbb{Z}_2 \) (\( r \geq 2 \)).

**Lemma 2.** Suppose there is a fixed point. Then the fundamental class \( U \) of \( H^n(X) \) survives in \( E^0_\infty \) and if \( u \in E^0_\infty \) is nontorsion with respect to \( H^*(B_G) \), there exists a \( v \in E^0_\infty \) such that \( uv = U \) (cup product).

Lemma 1 is proven by induction. 1 and 2 are true for \( G = S^1 \) since \( E_2 = H^*(B_G) \otimes H^*(X) \), and for \( G = \mathbb{Z}_p \) by known results of homological


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algebra (see [3]). The induction step is then shown by straightforward degree arguments.

Lemma 2 is proven by restriction to a $N$-dimensional orientable submanifold $B \subset B_G$ for large $N$. Then since $H^i(X, Z_p) = Z_p$ and $Z_p$ has no nontrivial action on $Z_p$, the local coefficients are trivial in the top dimension. Thus by piecing together over neighborhoods on which $X_G|_B$ is trivial, it is easy to show that $X_G|_B$ satisfies Poincaré duality with a cohomology fundamental class $[B]U$ where $X_G|_B$ is the portion of $X_G$ over $B$ and $[B]$ is the fundamental class of $B$. Using the fact that the inclusion $X_G|_B \to X_G$ induces an isomorphism on $E^j_{ij}$ for $j \leq N$ implies it induces an injection on $E^j_{ij}$ for $j \leq N$, and choosing $N$ large enough so it induces an isomorphism on $E^0_{i*}$, the lemma follows by finding a class in $H^*(X_G|_B)$ dual to $[B]u$.

With these two lemmas the proof of Bredon is valid without change.

2. Geometric proof. We shall assume that

(i) $X$ can be embedded in Euclidean space as a neighborhood retract.
(ii) $X$ has a finite number of orbit types.

Property (i) is inherited by the fixed point set, as follows from (ii) and the equivariant embedding theorem. Because of (i) $X$ is a Poincaré duality space over $K$ of formal dimension $n$ if and only if for any embedding $X \subset S^{n+r} = S$

\[ x \mapsto x \cup U : H^*(X) \to H^*(S, S - X) \]

for some $U \in H^*(S, S - X)$.

Choose a $G$-equivariant embedding, and use $K$ as the coefficient field. Then $H^*(S, S - X) = H^*_G(S, S - X)$ and we consider $U$ as an element of both groups, where we define for any $G$-pair $(A, A')$, $H^*_G(A, A') = H^*(A_G, A'_G)$. By induction on the cells in $B_G$, we see that there is an isomorphism

\[ \cup U : H^*_G(X) \to H^*_G(S, S - X). \]

Let $\Sigma$ be the fixed sphere in $S$ and $F = X \cap \Sigma$ the fixed set in $X$. Then the following diagram is commutative:

\[
\begin{array}{ccc}
H^*_G(S, S - X) & \xrightarrow{i^*} & H^*_G(\Sigma, \Sigma - F) \\
\cong \downarrow \cup U & & \cup i^*(U) \\
H^*_G(X) & \xrightarrow{i^*} & H^*_G(F)
\end{array}
\]

After localizing, the maps $i^*$ and hence all the maps in the diagram are isomorphisms. Localizing here means tensoring over $K[t, t^{-1}] \subset H^*(B_G)$ with $K[t, t^{-1}]$. The map on the right splits according to the connected
components of $F$, so we may assume $F$ is connected. Then

$$i^*(U) = t^n u_0 + t^{n-1} u_1 + \ldots + u_a + sv$$

where $u_i \in H^*(\Sigma, \Sigma - F)$, $u_0 \neq 0$, and $s = 0$ if $G = Z_2$ or $S^1$. Hence

$$\cup u_0 : H^*(F) \to H^*(\Sigma, \Sigma - F)$$

is an isomorphism, so $F$ is a Poincaré duality space over $K$.

References