COMPACT HILBERT CUBE MANIFOLDS AND THE INVARIANCE OF WHITEHEAD TORSION

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ABSTRACT. In this note we prove that every compact metric manifold modeled on the Hilbert cube \( Q \) is homeomorphic to \( |K| \times Q \), for some finite simplicial complex \( K \). We also announce an affirmative answer to the question concerning the topological invariance of Whitehead torsion for compact, connected CW-complexes. As a corollary of this latter result it follows that two compact Hilbert cube manifolds are homeomorphic iff their associated polyhedra (in the sense above) have the same simple homotopy type.

1. Introduction. In this note we announce some recent results on infinite-dimensional manifolds which imply, among other things, the topological invariance of Whitehead torsion for compact connected CW-complexes.

A Hilbert cube manifold (or \( Q \)-manifold) is a separable metric space which has an open cover by sets which are homeomorphic to open subsets of the Hilbert cube \( Q \). We say that a \( Q \)-manifold \( X \) can be triangulated (or is triangulable) provided that \( X \) is homeomorphic \( (\cong) \) to \( |K| \times Q \), for some countable locally-finite simplicial complex \( K \). In [5] it was shown that (1) any open subset of \( Q \) is triangulable, and (2) if \( X \) is any \( Q \)-manifold, then \( X \times [0,1) \) is openly embeddable in \( Q \) and thus is triangulable (where \( [0,1) \) is the half-open interval). We refer the reader to [4] for a list of earlier results on \( Q \)-manifolds. In this note, based on results in [6], we prove that every compact \( Q \)-manifold can be triangulated. The triangulation of non-compact \( Q \)-manifolds is much more delicate and is expected to be the subject of a future paper.

**Triangulation Theorem.** Every compact \( Q \)-manifold can be triangulated.

Using a result of West [13] it follows that if \( K \) is any finite simplicial complex, then \( |K| \times Q \cong M \times Q \), for some finite-dimensional combinatorial manifold \( M \). In this sense it follows that all compact \( Q \)-manifolds can be combinatorially triangulated.

**Corollary 1.** Every compact \( Q \)-manifold can be combinatorially triangulated.
This result contrasts sharply with the corresponding finite-dimensional situation, as there exist compact finite-dimensional manifolds which cannot be combinatorially triangulated [8].

In [13] West proved that if $K$ is any finite complex, simplicial or CW, then $|K| \times Q$ is a $Q$-manifold. Combining this result with the Triangulation Theorem we have the following result on finiteness of homotopy types.

**Corollary 2.** If $X$ is a compact metric space which is locally triangulable, then $X$ has the homotopy type of a finite complex.

In particular $X$ might be a compact $n$-manifold. Thus Corollary 2 strengthens the result of Kirby-Siebenmann on the finiteness of homotopy types of compact $n$-manifolds [9], but it is questionable whether this gives a simpler proof. It also sheds some light on the more general open question concerning the finiteness of homotopy types of compact ANR's [12].

The following result gives a topological characterization of simple homotopy types of finite CW-complexes in terms of homeomorphisms on $Q$-manifolds. We refer the reader to [7] for a proof.

**Characterization of Simple Homotopy Types.** Let $K$, $L$ be finite, connected CW-complexes and let $f: |K| \to |L|$ be a map. Then $f$ is a simple homotopy equivalence iff the map

$$f \times \text{id}: |K| \times Q \to |L| \times Q$$

is homotopic to a homomorphism of $|K| \times Q$ onto $|L| \times Q$.

We remark that the “only if” part of this theorem is just West's theorem on simple homotopy types [13]; i.e., if $f: |K| \to |L|$ is a simple homotopy equivalence, then $f \times \text{id}: |K| \times Q \to |L| \times Q$ is homotopic to a homeomorphism. As an immediate corollary of the above theorem we answer affirmatively the question on the topological invariance of Whitehead torsion [11, p. 378].

Waldhausen is reported to have an earlier and completely different proof of this Whitehead invariance problem.

**Corollary 3.** If $K$, $L$ are finite, connected CW-complexes and $f: |K| \to |L|$ is a homeomorphism, then $f$ is a simple homotopy equivalence.

We can also easily use the above Characterization to classify homeomorphic compact, connected $Q$-manifolds by simple homotopy type.

**Corollary 4.** Let $X$, $Y$ be compact, connected $Q$-manifolds and let

$$X \cong |K| \times Q, \quad Y \cong |L| \times Q$$

be any triangulations. Then $X \cong Y$ iff $|K|$ and $|L|$ have the same simple homotopy type.

Our proof we give here of the Triangulation Theorem and the proof of
the Characterization of Simple Homotopy Types, which is given in [7], rely mainly on the work in [6]. For our purposes here we use the following result from [6].

For notation $R^n$ denotes Euclidean $n$-space, $B^n$ denotes the standard $n$-ball of radius 1, $\text{Int}(B^n)$ denotes the interior of $B^n$, and $S^{n-1}$ denotes the boundary.

**Straightening Lemma.** If $X$ is a triangulated $Q$-manifold and $h: R^n \times Q \to X$ is an open embedding, for $n \geq 2$, then $X\setminus h(\text{Int}(B^n) \times Q)$ is a triangulated $Q$-manifold.

The proof of the Straightening Lemma given in [6] uses a considerable amount of infinite-dimensional machinery— influenced by finite-dimensional techniques. Some of the techniques used are infinite-dimensional surgery and infinite-dimensional handle straightening. These are just infinite-dimensional versions of some finite-dimensional techniques which were used in [9] for the recent triangulation results concerning $n$-manifolds.

Doing surgery on $Q$-manifolds is not nearly as difficult as it is on $n$-manifolds. In particular, the delicate inductive process of exchanging handles can always be done in two steps. Poincaré duality and transversality are never used. For straightening handles in $Q$-manifolds homeomorphisms on $T^n \times Q$ (where $T^n$ represents the $n$-torus) are used, much in the same way that torus homeomorphisms are used in finite-dimensional handle straightening. Also the theorem of West on simple homotopy type plays the role of the $s$-cobordism theorem.

In §2 we briefly describe some infinite-dimensional results which will be needed. Then in §3 we use these results, along with the Straightening Lemma, to prove the Triangulation Theorem. We remark that no prior experience with infinite-dimensional topology is needed for reading this note.

2. Infinite-dimensional preliminaries. We view $Q$ as the countable infinite product of closed intervals $[-1, 1]$ and we let 0 represent the point $(0, 0, \ldots)$ of $Q$. Most basic is Anderson's notion of Property Z [1]. A closed subset $A$ of a space $X$ is said to be a $Z$-set in $X$ provided that for each nonnull and homotopically trivial open subset $U$ of $X$, $U \setminus A$ is also nonnull and homotopically trivial. A map $f : X \to Y$ (i.e., a continuous function) is said to be a $Z$-embedding provided that $f$ is a homeomorphism of $X$ onto a $Z$-set in $Y$. We now state two properties of $Z$-sets in $Q$-manifolds which will be needed in §3. For details see [2] and [5].

**Approximation Lemma.** Let $A$ be a compact metric space, $X$ be a $Q$-manifold, and let $f : A \to X$ be a map. Then $f$ is homotopic to a $Z$-embedding $g : A \to X$. In fact, $g$ can be chosen arbitrarily close to $f$. 

COLLARING LEMMA. Let $K$ be a finite complex, $X$ be a $Q$-manifold, and let $f:|K| \times Q \to X$ be a $Z$-embedding. Then there exists an open embedding $g:|K| \times Q \times [0,1) \to X$ such that $g(k,q,0) = f(k,q)$, for all $(k,q) \in |K| \times Q$.

We will also need the following result on the characterization of $Q$. For details see [5].

CHARACTERIZATION OF $Q$. If $X$ is a compact contractible $Q$-manifold, then $X \cong Q$.

3. Proof of the Triangulation Theorem. Without loss of generality, let us consider a compact connected $Q$-manifold $X$ which we describe to triangulate. Since $X$ is a compact ANR (metric), it follows that $X$ is dominated by a finite complex (see [3, p. 106]). Thus $\pi_1(X)$ and $H_*(X)$ are finitely generated, where we use singular homology with integral coefficients. By a standard process, all of the homotopy groups of $X$ can be killed by attaching a finite number of $n$-cells to $X$, for $n \geq 2$. One uses the fact that $\pi_1(X)$ is finitely generated to kill $\pi_1(X)$, and after this the Hurewicz isomorphism theorem is used to inductively kill the higher homotopy groups. Since $H_*(X)$ is finitely generated it terminates after a finite number of cell attachments. We omit the details. Thus we obtain a finite sequence $X = X_0, X_1, \ldots, X_p$ of spaces such that each $X_i$ is obtained by attaching some $n$-cell to $X_{i-1}$, and $\pi_n(X_p) = 0$ for all $n$. Since $X_p$ is an ANR, it must be contractible (see [3, p. 124]). Of course the spaces $X_i$, $1 \leq i \leq p$, are not $Q$-manifolds. We are going to modify the above procedure to obtain a sequence $X = X'_0, X'_1, \ldots, X'_p$ of compact $Q$-manifolds such that each $X'_i$ has the homotopy type of $X_i$. We construct this sequence inductively.

For each $j$, $0 \leq j \leq p$, let $S_j$ be the following statement: There exists a sequence $X = X'_0, X'_1, \ldots, X'_j$ of compact $Q$-manifolds such that (1) for $0 \leq i \leq j$, $X'_i$ has the homotopy type of $X_i$, and (2) for $1 \leq i \leq j$, if $X'_i$ is triangulable, then so is $X'_{i-1}$. It is clear that $S_0$ is true. Thus assume that $S_j$ is true, for some $j < n$, and let $X = X'_0, X'_1, \ldots, X'_j$ be the corresponding sequence of $Q$-manifolds satisfying (1) and (2). Let $\alpha: X_j \to X'_j$ be a homotopy equivalence and let $f:S^{n-1} \to X_j$ be the attaching map used to construct $X_{j+1}$; i.e., $X_{j+1} = X_j \cup_f B^n$. Define $f':S^{n-1} \times Q \to X'_j$ by $f'(x,q) = \alpha f(x)$, for all $(x,q) \in S^{n-1} \times Q$, and using the Approximation Lemma let $g:S^{n-1} \times Q \to X'_j$ be a $Z$-embedding which is homotopic to $f'$. We define $X'_{j+1} = X'_j \cup_k (B^n \times Q)$, the space formed by attaching $B^n \times Q$ to $X'_j$. It easily follows from the Collaring Lemma that $X'_{j+1}$ is a $Q$-manifold. In fact, there exists an open embedding $h:R^n \times Q \to X'_{j+1}$ such that $X'_j \cong X'_{j+1} \setminus h(\text{Int}(B^n) \times Q)$. Applying the Straightening Lemma it follows that if $X'_{j+1}$ is triangulable, then so is $X'_j$. All we need to do now
is check that $X'_{j+1}$ and $X_{j+1}$ have the same homotopy type. Let $g': S^{n-1} \to X'_j$ be defined by $g'(x) = g(x, 0)$, for all $x \in S^{n-1}$. Since $Q$ contracts to $0 \in Q$, it follows that $X'_{j+1}$ has the homotopy of $X_j \cup_k B^n$. But $g'$ is homotopic to $g : S^{n-1} \to X_j$. Using Theorem 2.3 on p. 120 of [10], it follows that $X_j \cup_k B^n$ has the homotopy type of $X_{j+1}$. Thus $S_{j+1}$ is true.

Since $S_p$ is true we have a sequence $X = X'_0, X'_1, \ldots, X'_p$ of compact $Q$-manifolds such that $X'_i$ is contractible and if $X'_i$ is triangulable, then so is $X_{i-1}'$, for $1 \leq i \leq p$. Using the Characterization of $Q$ it follows that $X'_p \simeq Q$, which is triangulable. Then inductively working our way back down to $X$ it follows that $X$ is triangulable.

REFERENCES


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