TRACE CLASS, WIDTHS AND THE FINITE APPROXIMATION PROPERTY IN BANACH SPACE

BY R. A. GOLDSTEIN AND R. SAEKS

Communicated by Robert G. Bartle, June 21, 1972

I. Introduction. Although there have been a number of attempts [3] to define operator classes in Banach space whose properties are analogous to the classical trace class operators of Hilbert space [1], [2] it is generally agreed that a satisfactory definition has yet to be achieved [3]. The purpose of the present note is to introduce a new approach to the problem wherein operator widths [2], [4] in Banach space replace the eigenvalues of the Hilbert space formulation; the viability of the approach being illustrated by the formulation of a number of sufficient conditions for an operator to have the finite approximation property in terms of its widths. Moreover, unlike the previous approaches [3] the trace class operators defined via operator widths are representation independent and coincide exactly with the classical definitions in Hilbert space.

II. Definitions and results. In the sequel $X$ is a Banach space normed by $\| \cdot \|$, $B$ is the unit ball in $X$ and $\mathcal{P}_n$ is the set of $n$-dimensional subspaces of $B$. The $n$th width, $d_n(A)$, of an operator $A$ on $X$ is defined [4] by

$$d_n(A) = \inf_{L \in \mathcal{P}_n} \sup_{v \in L} \inf_{u \in \mathcal{B}} \| Au - v \|.$$

Classically, Kolmogorov [4] defined the $n$th width of a set to be a measure of the degree to which the set could be approximated by $n$-dimensional subspaces, the definition of equation (1) being that of Kolmogorov applied to $A(B)$.

Some remarks concerning the sequence $\{d_n(A)\}$ are as follows:

(a) $d_0(A) = \|A\|$;

(b) $\{d_n(A)\}$ is a nonincreasing sequence and $d_n(A) \to 0$ iff $A$ is a compact operator;

(c) (see for example [2]) for $X$ a Hilbert space and $A$ a compact linear operator on $X$, set $s_n(A) \equiv \lambda_n((A*A)^{1/2}) \equiv$ the $n$th eigenvalue of $(A*A)^{1/2}$ ($n = 1, 2, \ldots$) (these are called the $s$ numbers or characteristic numbers of $A$). Then $d_n(A) = s_{n+1}(A)$ ($n = 0, 1, 2, \ldots$). $A$ is an Hilbert-Schmidt or nuclear operator if the sequence $\{s_n(A)\}$ is an $l_2$ or $l_1$ sequence.


Key words and phrases. Widths, trace-class, finite approximation property, $d$-nuclear, compact operator.
We define the “trace” classes $D_p$ of compact linear operators on a Banach space as

$$D_p = \left\{ A \text{ compact, linear} \left| \sum_{n=0}^{\infty} d_n(A)^p < \infty \right. \right\}.$$  

(In the case of a Hilbert space they reduce to the classes $C_p$ or $S_p$ defined in [1] or [2].)

Define $\|.|:D_p \rightarrow R$ by

$$\|A\|_p = \left[ \sum_{n=0}^{\infty} d_n(A)^p \right]^{1/p}.$$  

**Theorem 1.** $\|A\|_p$ is a norm on $D_p$.

An operator $A \in D_1$ will be called $d$-nuclear.

Before proceeding to our main theorem on the approximation of trace class operators, we introduce another sequence, $p_n(X)$, of positive numbers — which characterize the Banach space $X$ rather than the operator $A$ defined by

$$p_n(X) = \sup_{L: \mathcal{L}_n(L) \subseteq X} \inf_{P(L) \in P(L)} \|P(L)\|,$$

where $\mathcal{L}_n$ denotes the set of all $n$-dimensional linear subspaces of $X$, and $P(L)$ denotes the set of all projections into the subspace $L$.

(i) $1 \leq p_n(X) \leq n$.

(ii) If $X$ is a Hilbert space, $p_n(X) \equiv 1 \ (\forall n)$.

(iii) $p_n(X) = p_n(X^*)$ if $X$ is reflective.

(iv) Murray [5] has shown that $p_n(X)$ grows linearly with $n$ for $X = L_p$ or $l_p \ (p \neq 2)$.

We say that $A$ has the finite approximation property iff $\exists$ a sequence of operators, $A_n$, with finite-dimensional ranges, such that $\|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$.

**Theorem 2.** Let $A$ be a compact operator on $X$ such that

$$\lim_{n \rightarrow \infty} [d_n(A)p_n(X)] = 0.$$

Then $A$ has the finite approximation property.

Since, in Hilbert space $p_n(X) = 1$, and $d_n(A) \rightarrow 0$ for $A$ compact, we trivially obtain the classical result

**Corollary 3.** Every compact operator on a Hilbert space has the finite approximation property.
Although we are not able to answer the classical conjecture on the finite approximation property for any compact operator on a Banach space we can observe

**Corollary 4.** Every \( A \in D_1 \) has the finite approximation property.

This holds because \( \{d_n(A)\} \in l_1 \), hence \( d_n(A) = O(1/n^{1+\epsilon}) \) (\( \epsilon > 0 \)) and \( d_n(A)p_n(x) = O(n^{-1}/n^{1+\epsilon}) = O(n^{-\epsilon}) \) (\( n \to \infty \)).

The standard examples of nuclear operators in Hilbert space have natural analogs in Banach space to which the corollary applies. In particular, the injection maps of Sobolev spaces \( W_0^{1, p} \) into \( L^p \) is \( d \)-nuclear, and integral operators with smooth kernels on \( L^p \) into \( L^p \) are \( d \)-nuclear.

**Proof of Theorem 2.**

\[
d_n(A) = \min \max_{L \in \mathcal{L}_n} \|Ax - L\| = \min \max_{L \in \mathcal{L}_n} \max_{x \in L^1; \|x\| = 1} \|x^*Ax\|
\]

\[
= \min \max_{L \in \mathcal{L}_n, x^* \in L^1; \|x^*\| = 1} \|A^*x^*\|, \text{ where } X^* \supset L^1 \text{ is the annihilator of } L.
\]

**Lemma.** \( P(L)^* = 1 - P(L^1). \)

Let \( \tilde{L} \in \mathcal{L}_n \) be the minimizing subspace for \( d_n(A) \) (which exists since \( A \) is compact). Then \( \exists \) a projection \( P_n = P(\tilde{L}) \) such \( \|P_n\| = O(p_n(X)) \). Set \( P^\perp_n \equiv P(\tilde{L}^\perp). \)

We then have

\[
d_n(A) = \sup_{x^* \in X^*} \frac{\|A^*P^\perp_n x^*\|}{\|P^\perp_n x^*\|} \leq \sup_{x^* \in X^*} \frac{\|A^*P^\perp_n x^*\|}{\|P^\perp_n\| \|x^*\|}.
\]

Since \( \|P_n\| = \|1 - P_n\| \leq 1 + \|P_n\| \), we have

\[
2p_n(X)d_n(A) \geq \sup_{x^*} \|A^*P^\perp_n x^*\|/\|x^*\| = \|A^*P^\perp_n\| = \|(1 - P_n)A\|.
\]

Setting \( A_n \equiv P_nA, \|A - A_n\| = O(d_n(A)p_n(X)) \) and the theorem is proved.

We conjecture that the condition of Theorem 2 is also necessary.

**Acknowledgement.** The authors wish to express their gratitude to Professor Abraham Goetz whose insights and suggestions help to guide and formulate their work.

**References**


DEPARTMENT OF MATHEMATICS AND THE DEPARTMENT OF ELECTRICAL ENGINEERING, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556