A HOMOTOPY CLASSIFICATION OF 2-COMPLEXES WITH FINITE CYCLIC FUNDAMENTAL GROUP

BY MICHEAL N. DYER AND ALLAN J. SIERADSKI
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For an arbitrary positive integer $n$, let $Z_n$ denote the cyclic group of order $n$, and let $P_n = S^1 \cup_n e^2$ be the pseudo-projective plane of order $n$.

**Theorem.** Let $X$ be a connected finite 2-dimensional CW-complex with fundamental group $Z_n$. Then

1. $X$ has the homotopy type of the sum $P_n \lor S^2 \lor \cdots \lor S^2$ of the pseudo-projective plane $P_n$ and rank $H_2(X)$-copies of the 2-sphere $S^2$.

2. There is a homotopy equivalence $f: X \to P_n \lor S^2 \lor \cdots \lor S^2$ realizing any prescribed Whitehead torsion $\tau(f) \in Wh(Z_n)$.

The result (1) was established in the prime order case by W. H. Cockcroft and R. G. Swan [3]. The work of P. Olum on the self-equivalences of the pseudo-projective plane $P_n$ ([6], [7]) shows that every element of the Whitehead group $Wh(Z_n)$ is realized as the torsion of some self-equivalence $P_n \to P_n$, so that (2) is a consequence of (1).

**Corollary.** For connected finite 2-dimensional CW-complexes with finite cyclic fundamental group, homotopy type and simple homotopy type coincide.

This generalizes to the nonprime order case a recent observation of W. H. Cockcroft and R. M. F. Moss [2].

**Sketch of a Proof of the Theorem.** Each CW-complex under consideration has the simple homotopy type of a complex $P$ that is modeled in an obvious fashion on some presentation $\mathcal{P} = \langle a_1, \ldots, a_k : r_1, \ldots, r_m \rangle (m \geq k)$ of the cyclic group $Z_n$. There are Nielsen transformations which reduce such a presentation to one of pre-Abelian form [5, p. 140]

$$\mathcal{Q} = \langle b_1, \ldots, b_k : b_1 W_1, \ldots, b_k W_k, W_{k+1}, \ldots, W_m \rangle,$$
where the exponent sum of each word $W_i$ with respect to each generator $b_j$ is zero. Moreover, this Nielsen reduction $\mathcal{P} \to \mathcal{Q}$ corresponds to a simple homotopy equivalence $P \to Q$ of the associated topological models. Associated with each topological model $P$ of a presentation $\mathcal{P}$ is the cellular chain complex $C_*(P)$ of its universal covering $\tilde{P}$; the chain groups are free $Z_n$-modules which we give preferred bases according to a

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specific natural system. The chain complex \( C_\ast = C_\ast(\tilde{Q}) \) with its preferred bases is

\[
\begin{array}{cccc}
C_2(\tilde{Q}) & C_1(\tilde{Q}) & \partial_1 & C_0(\tilde{Q}) \\
\{u_1, \ldots, u_m\} & \delta_2 & \{v_1, \ldots, v_k\} & (0, \ldots, 0, x-1) \\
\end{array}
\]

where \( \{u_1, \ldots, u_m\} \) is the free \( \mathbb{Z}_n \)-module with the enclosed basis, and \( x \) is the generator of the multiplicative cyclic group \( \mathbb{Z}_n \).

Using Jacobinski's cancellation theorem for projective \( \mathbb{Z}_n \)-modules \([4, 8, p. 215], [9, p. 178]\), it is possible to choose a new basis \( w_1, \ldots, w_m \) for the chain group \( C_2 = C_2(\tilde{Q}) \) such that the matrix of the boundary operation \( \partial_2: C_2(\tilde{Q}) \to C_1(\tilde{Q}) \) with respect to this new basis for \( C_2 \) and the old basis \( v_1, \ldots, v_k \) for \( C_1 \) is

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & N & 0 & 0
\end{pmatrix}
\]

where the identity block is a \( (k-1) \times (k-1) \) matrix and where \( N = 1 + x + \cdots + x^{n-1} \) is in the integral group ring of \( \mathbb{Z}_n \). The chain complex \( C_\ast \) with the new preferred basis for \( C_2 \) takes the form

\[
\begin{array}{cccc}
C_2 & C_1 & C_0 \\
\{w_1, \ldots, w_m\} & \delta_2 & \{v_1, \ldots, v_k\} & (0, \ldots, 0, x-1) \\
\{u_1, \ldots, u_m\} & \partial_1 & \{v_1, \ldots, v_k\} & (0, \ldots, 0, x-1) \\
\end{array}
\]

With these preferred bases, the chain complex \( C_\ast \) is realizable as the cellular chain complex \( C_\ast(\tilde{R}) \) of the universal covering \( \tilde{R} \) of the complex \( R \) modeled on the presentation \( \mathcal{R} = \langle c_1, \ldots, c_k : c_1, \ldots, c_k-1, c_k^n, 1, \ldots, 1 \rangle \) with \( m - k \) trivial relators. The identity map between the chain complexes \( C_\ast(\tilde{R}) \) and \( C_\ast(\tilde{Q}) \) can be realized by a map \( f: R \to Q \) that is necessarily a homotopy equivalence. This completes the proof of the theorem since the space \( R \) modeled on the presentation \( \mathcal{R} \) has the simple homotopy type of the sum \( P_n \vee S^2 \vee \cdots \vee S^2 \) of the pseudo-projective plane \( P_n \) and \( m - k \) copies of the 2-sphere \( S^2 \).

Full details of these and related results will appear elsewhere.

**BIBLIOGRAPHY**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403