THE MORSE LEMMA ON ARBITRARY BANACH SPACES

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Communicated by Everett Pitcher, May 19, 1972

In [4], the author proved the Morse lemma on a real Banach space $E$ which is the dual space of some space $E_0$, as for example the Sobolev spaces $L_p^k, k \geq 0, 1 < p < \infty$ and the Hölder spaces $C^{k,\alpha}$ [5]. The author’s first result extended earlier versions by Morse and Palais [2], [3]. In this note we state a theorem and sketch a proof of the Morse lemma for any Banach space.

Let $f : U \to \mathbb{R}$ be at least $C^3$ (3 times differentiable) with $0 \in U$ a critical point of $f$ ($Df_0 = 0$). By the Taylor theorem we can write $f$ as

$$f(x) = \frac{1}{2} \langle A_x x, x \rangle + f(0)$$

where $A : U \to L(E, E^*)$ (the linear maps from $E$ to $E^*$) is $C^1$ and symmetric; i.e.,

$$\langle A_x u, v \rangle = \langle A_x v, u \rangle \quad \forall u, v \in E.$$

Here $\langle A_x u, v \rangle$ denotes the standard bilinear pairing of $E$ and $E^*$.

DEFINITION. 0 is said to be a nondegenerate critical point if

1. There exists a nbhd $N \subset U$ of 0 and constants $C_1$ and $C_2$ so that $\forall t, t', t_1, t_2 \in N$.
   a. $A^*_t$ is injective (thus $A_t$ is injective).
   b. $\|DA_t(y)\| \leq C_1 \|h\| \cdot \|A_t y\|$ for all $h, y \in E$.
   c. $\|DA_{t_1}(h)(y) - DA_{t_2}(h)(y)\| \leq C_2 \|h\| \cdot \|t_1 - t_2\| \cdot \|A_{t_1} y\|$ for all $h, y \in E$, where $D$ denotes the Fréchet derivative of $A$ with respect to the subscript variable.

2. (a) For each $t \in N$, $\langle A_{t} x_n, y \rangle$ converges to zero for all $y$ iff $\langle A_{0} x_n, y \rangle$ converges to zero for all $y$.
   (b) Given $t \in N$ if $\langle A_{t} x_n, y \rangle$ converges to zero for all $y \in E$ then $\langle DA_{t}(h)(x_n), y \rangle$ converges to zero of all $y \in E$ and $h \in E$.

It is not difficult to check that if $E = H$ (Hilbert space) and $A_0 : H \to H$ is an isomorphism (the standard definition of nondegeneracy) then conditions (1) and (2) are satisfied.

THEOREM (MORSE LEMMA). Let $f : U \to \mathbb{R}$ be $C^3$ with $0 \in U$ a nondegenerate critical point of $f$. Then there exists a local diffeomorphism $\phi$ of a nbhd of 0 so that

$$f \circ \phi(x) = \frac{1}{2} D^2 f_0(x, x) + f(0),$$

where $D^2 f_0$ is the second derivative of $f$ at 0.

We shall sketch the principal idea in the proof. Let $t \in N$ where $N$ is
given as above. For each $y \in E$ define $f_y \in (\text{Range } A_t)^*$ by $f_y(A_t x) = \langle A_0 x, y \rangle$. Condition 2(a) guarantees that $f_y$ is continuous on $\text{Range } A_t$.
Condition 1(a) says that $\text{Range } A_t = E^*$. Thus 1(a) and 2(a) together imply that $f_y$ can be extended uniquely to an element of $E^{**}$.

Let $\Gamma$ be the set of functionals on $E^*$ induced by $E$ (the weak * topology). Condition 2(a) says that $f_y$ is $\Gamma$-continuous for each $y$ and therefore by a 
well-known result (cf. [1, p. 420]) in functional analysis $f_y \in \Gamma$. Thus there is an element $Q_t y \in E$ so that

$$\langle A_t x, Q_t y \rangle = \langle A_0 x, y \rangle \text{ or } \langle A_t Q_t y, x \rangle = \langle A_0 y, x \rangle \quad \forall x, y.$$

This implies that $A_t Q_t = A_0$. From conditions 1(b) and 1(c) and 2(b) one can show that $Q_t \in GL(E)$ and that $t \to Q_t$ is $C^1$. Now $Q_0 = I$ and so locally we get a map $P_t \in GL(E)$, $t \to P_t$, $C^1$ with $P_t^2 = Q_t$. Set $R_t = P_t^{-1}$. As in [4] it follows that $\langle A_t u, v \rangle = \langle A_0 R_t u, R_t v \rangle$, $\forall u, v \in E$. From the inverse function theorem it follows that the map $\psi : x \to R_t(x)$ has a local inverse $\phi$ and that for

$$f(t) = \frac{1}{2} \langle A_0(t), (t) \rangle + f(0)$$

$$= \frac{1}{2} \langle A_0 R_t(t), R_t(t) \rangle + f(0),$$

so that

$$f \circ \phi(t) = \frac{1}{2} \langle A_0 t, t \rangle + f(0)$$

$$= \frac{1}{2} D^2 f_0(t, t) + f(0)$$

which concludes the proof.

**BIBLIOGRAPHY**