A CHARACTERIZATION OF GROWTH IN LOCALLY COMPACT GROUPS

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$G$ will denote throughout a separable, connected, locally compact group. Fix a left Haar measure on $G$ and for a measurable subset $A$ of $G$, let $|A|_G$ denote the measure of $A$. The purpose of this note is to announce results concerning the asymptotic behavior of $|U^n|_G$ where $U$ is a compact neighborhood of the identity $e$ in $G$, and to indicate some of the applications these results have for various areas. The following definitions are required:

**Definition 1.** $G$ has **polynomial growth** if there is a polynomial $p$ such that for each compact neighborhood $U$ of $e$, there is a constant $C(U)$ so that

$$|U^n|_G \leq C(U)p(n) \quad (n = 1, 2, \ldots)$$

($U^n = \{u_1 u_2, \ldots, u_n | u_i \in U, 1 \leq i \leq n\}$). $G$ has **exponential growth** if for each compact neighborhood $U$ of $e$ there is a $t > 1$ such that

$$|U^n|_G \geq t^n \quad (n = 1, 2, \ldots).$$

Note that since $G$ is connected, its “growth” will be determined by the behavior of $|U^n|_G$ for any one compact neighborhood $U$ of $e$.

For $a, b \in G$, let $[a, b]$ denote the subsemigroup of $G$ generated by $a$ and $b$, i.e.,

$$[a, b] = \{x_1 x_2, \ldots, x_n | x_i \in \{a, b\}, 1 \leq i \leq n, n = 1, 2, \ldots\}.$$ 

$[a, b]$ is said to be free if $a[a, b] \cap b[a, b] = \emptyset$. A subset $S$ of $G$ is uniformly discrete if there is a neighborhood $U$ of $e$ in $G$ such that $sU \cap tU = \emptyset$ for $s, t \in S, s \neq t$.

**Definition 2.** $G$ is type $NF$ if there does not exist $a, b \in G$ such that $[a, b]$ is free and uniformly discrete.

Let $H$ be a connected Lie group with Lie algebra $\mathfrak{h}$, and let $g \to \text{Ad} g$ be the canonical adjoint representation of $H$ on $\mathfrak{h}$. $H$ is said to be type $R$ if the eigenvalues of $\text{Ad} g$ are of absolute value one for each $g \in H$.

Since $G$ is connected, there exists an arbitrarily small compact normal subgroup $K$ of $G$ such that $G/K$ is a Lie group.

**Definition 3.** $G$ is type $R$ if there exists a compact normal subgroup $K$
such that \( G/K \) is a type \( R \) Lie group.

**Theorem 4.** The following conditions are equivalent:

(i) \( G \) has polynomial growth,

(ii) \( G \) is type \( NF \),

(iii) \( G \) is type \( R \).

**Outline of Proof.** (i) \( \Rightarrow \) (ii) is straightforward. To establish that (ii) \( \Rightarrow \) (iii), we define groups \( G_\theta \) for each \( \theta = \theta_1 + i\theta_2, \theta_1, \theta_2 \in \mathbf{R}, \theta_1 \neq 0 \) and show that each \( G_\theta \) is not type \( NF \) and that, if \( G \) is not type \( R \), \( G \) contains some \( G_\theta \) as a topological subgroup. It then follows that \( G \) is not type \( NF \).

To show that (iii) \( \Rightarrow \) (i), we first reduce to the case where \( G \) is simply connected and solvable. One can then write \( G = g_1(t_1)g_2(t_2)\cdots g_n(t_n) \) where each \( g_i(t_i) \) is a one parameter subgroup of \( G \). The argument proceeds by induction on \( n \), using the fact that since \( G \) is type \( R \), \( ||\text{Ad}g(t)||^p(t) \) for some polynomial \( p \).

**Comparison with discrete groups.** Milnor [8] and Wolf [11] have investigated the growth of discrete solvable groups in connection with the study of fundamental groups of Riemannian manifolds with negative curvature. Combining their results with a recent result of Tits [10], one has the following: If \( H \) is a linear group over a field \( k \) with a finite set of generators \( A = A^{-1} \), then (i) either \( |A^n|_H \leq p(n) \) for some polynomial \( p \) and all \( n \geq 1 \) or there is a \( t > 1 \) such that \( |A^n|_H \geq t^n \) for all \( n \geq 1 \), and (ii) if \( |A^n|_H \) has polynomial growth, then \( H \) is a finite extension of a solvable group \( S \) and \( S \) is a finite extension of a nilpotent group. We obtain analogous results for connected groups as a corollary to Theorem 4.

**Corollary 5.** (i) Either \( G \) has polynomial growth or \( G \) has exponential growth.

(ii) If \( G \) is a connected Lie group with polynomial growth, then \( G \) is the compact extension of a solvable Lie group \( S \) and \( \text{Ad} S \) is an analytic subgroup of a compact extension of a nilpotent group.

**Remark.** The first part of this corollary shows that in a connected group, a compact set cannot grow at a rate intermediate to polynomial and exponential, for example, such as \( t^n/\log n \). This answers a question raised in Emerson and Greenleaf [4]. With regard to the second part, we remark that Hulanicki [5] has shown that a separable, locally compact group that is the compact extension of a nilpotent group cannot have exponential growth.

**Strong amenability.** In [4] Emerson and Greenleaf define a locally compact group \( H \) to be strongly amenable if for every compact neighborhood \( U = U^{-1} \) of \( e \) in \( H \).
Greenleaf has asked if every connected, amenable, unimodular group is necessarily strongly amenable. The following corollary to Theorem 4 provides a large class of counterexamples.

**Corollary 6.** If $G$ is strongly amenable, then $G$ is type R. If $G$ is type R, then

$$\liminf_{n} \frac{|U^{n+1}|_H}{|U^n|_H} = 1$$

for each compact neighborhood $U$ of $e$.

In particular, let $G$ be the semidirect product of $\mathbb{R}$ with $\mathbb{R}^2$ given by the homomorphism $\varphi: \mathbb{R} \to \text{Aut}(\mathbb{R}^2)$ where $\varphi(t)(x, y) = (e^{tx}, e^{-ty})$ for $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$. Then $G$ is connected, amenable, unimodular but not type R, and hence, not strongly amenable.

**An ergodic theorem.** Let $X$ be a compact, separable metric space and assume $G$ is unimodular and has a jointly continuous action $G \times X \to X$ on $X$. A sequence of Borel subsets $\{A_n\}$ of $G$ is called balanced with respect to the action of $G$ on $X$ if $0 < |A_n|_G < \infty$ for each $n$ and if whenever $\mu$ is a probability measure on $X$ invariant and ergodic under $G$ and $f \in C(X)$, the continuous complex valued functions on $X$, then

$$\lim_{n} \frac{|A_n|_G^{-1} \int_{A_n} f(g \circ x_0) \, dg}{\int_{A_n} f(g \circ x_0) \, dg}$$

exist and equals $\int f \, d\mu$ for $\mu$-almost all $x_0 \in X$.

An increasing sequence of subsets $\{A_n\}$ of $G$ grows evenly in $G$ if $0 < |A_n|_G < \infty$ for each $n$,

$$\lim_{k} |A_k|_G^{-1} (A_k A_n) \Delta A_k|_G = 0 = \lim_{k} |A_k|_G^{-1} (A_n A_k) \Delta A_k|_G$$

for each $n$, and there is a constant $c > 0$ such that $|A_n^{-1} A_n|_G \leq c |A_n|_G$ for each $n$.

Calderón [3] and Bewley [2] have proved the following generalization of Birkhoff's individual ergodic theorem: If $G$ contains a sequence $\{A_n\}$ that grows evenly in $G$, then $\{A_n\}$ is balanced with respect to the action of $G$ on $X$.

Auslander and Brezin [1] have shown that any connected, simply connected, nilpotent Lie group $N$ contains a sequence of compact connected subsets that grow evenly in $N$. This is a special case of

**Corollary 7.** If $G$ satisfies the equivalent conditions of Theorem 4 and
if $U = U^{-1}$ is a compact neighborhood of the identity, then a subsequence of $\{U^n| n = 1, 2, \ldots\}$ grows unevenly in $G$.

**On symmetry of $\mathcal{L}^1(G)$**. A Banach $*$-algebra $\mathcal{U}$ is symmetric if $-xx^*$ is quasi-regular for each $x \in \mathcal{U}$, or equivalently by Raikov's Theorem [9], if

$$v(x) = \lim_{n} ||x^n||^{1/n} = \sup_x T_x$$

for each $x = x^* \in \mathcal{U}$, where the sup is taken over all $*$-representations $x \to T_x$ of $\mathcal{U}$. Hulanicki [5] has shown that if $H$ is a separable, locally compact group such that $\lim_n |A^n|^{1/n} \leq 1$ for any compact subset $A$ of $G$, then $v(x) = \lambda(x)$ for all $x = x^* \in \mathcal{L}^1(H)$ with compact support. Thus, any group with polynomial growth “almost” has a symmetric group algebra. (Observe that symmetry fails in this case only when the spectral radius is not continuous, and it is not known if this can ever occur in a group algebra.)

On the other hand, if $H$ is a discrete group, $L^1(H)$ is not symmetric if $H$ contains a free semigroup $[a, b]$ (cf. Jenkins [6]). There is evidence that suggests a similar statement obtains if $G$ is not type $NF$. Theorem 4, therefore, lends support to a conjecture this author originally stated in [7], to wit, $\mathcal{L}^1(G)$ is symmetric if, and only if, $G$ is type $NF$.

Proofs of these and related results will appear elsewhere. This author wishes to express his thanks to R. Howe for many helpful suggestions related to this work.

**REFERENCES**


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