PONTRJAGIN CLASSES OF PL SHEAVES

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ABSTRACT. Over the category of PL manifolds there is a fibered category whose objects are certain equivalence classes \([\mathcal{F}]\) of "PL sheaves" \(\mathcal{F}\), to which one assigns real characteristic classes as in [2] and [3]. In particular each PL manifold \(M\) possesses a distinguished (co)tangent object \([\mathcal{E}(M)]\) and a real Pontrjagin class \(p([\mathcal{E}(M)])\). In this note we show that \(p([\mathcal{E}(M)])\) is the image under \(H^{4*}(M;\mathbb{Q}) \to H^{4*}(M;\mathbb{R})\) of the Thom-Pontrjagin class of \(M\).

The construction of [2] and [3] assigns total Chern classes \(c([\mathcal{F}]) \in H^{2*}(M;\mathbb{R})\) to cosets \([\mathcal{F}]\) of complex PL sheaves \(\mathcal{F}\) over a PL manifold \(M\), and this assignment satisfies certain axioms. As in the classical case one defines the total Pontrjagin class \(p([\mathcal{F}]) \in H^{4*}(M;\mathbb{R})\) of a coset \([\mathcal{F}]\) of real PL sheaves via complexification of \([\mathcal{F}]\), and here are the corresponding axioms:

\[(P_1)\quad\text{if } [\mathcal{F}] \text{ is a coset of real PL sheaves of "rank" } m \text{ on a PL manifold } M \text{ then the total Pontrjagin class } p([\mathcal{F}]) \text{ is an element } 1 + p_1([\mathcal{F}]) + \cdots + p_{[m/2]}([\mathcal{F}]) \text{ of } H^{4*}(M;\mathbb{R}) \text{ with } p_1([\mathcal{F}]) \in H^{4}(M;\mathbb{R});\]

\[(P_2)\quad p(\Xi([\mathcal{F}])) = \Xi^* p([\mathcal{F}]) \in H^{4*}(N;\mathbb{R}) \text{ for any PL map } \Xi : N \to M;\]

\[(P_3)\quad p([\mathcal{F}] \oplus [\mathcal{G}]) = p([\mathcal{F}]) \cup p([\mathcal{G}]) \text{ for any cosets } [\mathcal{F}] \text{ and } [\mathcal{G}] \text{ over } M;\]

\[(P_4)\quad\text{if } [\mathcal{F}] \text{ contains a bona fide real vector bundle } \xi \text{ over } M \text{ (as in [2] or [3]) then } p([\mathcal{F}]) \text{ is the classical total Pontrjagin class } p(\xi) \in H^{4*}(M;\mathbb{R}).\]

**Lemma 1.** If a PL manifold \(M\) happens to admit a smooth structure with tangent bundle \(\tau_M\) then \(p([\mathcal{E}(M)]) = p(\tau_M).\)

**Proof.** One easily verifies as in [2] that \([\mathcal{E}(M)]\) contains \(\tau_M\); hence it suffices to apply \((P_4)\).

As in the smooth case one uses axioms \((P_1), (P_2), (P_3)\) and the multiplicative sequence corresponding to \(z^{1/2}/(\tanh z^{1/2})\) to construct the Hirzebruch polynomial \(l([\mathcal{F}]) \in H^{4*}(M;\mathbb{R})\) of the Pontrjagin class \(p([\mathcal{F}]),\)

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and one defines the $L$-genus $L(M)$ of any compact oriented PL manifold $M$ of dimension $4n$ by setting $L(M) = \langle l_n([\partial(M)]) \rangle$ for the fundamental class $[M] \in H_{4n}(M; \mathbb{R})$. The index $I(M)$ is defined as usual, and the classical Hirzebruch index formula combines with Lemma 1 to guarantee that $L(M) = I(M)$ whenever the PL manifold $M$ happens to have a smooth structure.

**Lemma 2.** $L(M) = I(M)$ for any compact oriented PL manifold $M$ of dimension $4n$.

**Proof.** One easily verifies as in the smooth case that

$$L(M + N) = L(M) + L(N), \quad L(-M) = -L(M), \quad L(M \times N) = L(M) \cdot L(N)$$

and $L(\text{boundary}) = 0$, so that $L$ may be regarded as a homomorphism $\Omega^*_e \otimes R \to R$. Hence it will suffice to verify that the homomorphism $L$ agrees on generators with the corresponding homomorphism $I : \Omega^*_e \otimes R \to R$. But the homomorphism $\Omega^*_e \to \Omega^*_e$ is injective (by (P_4), for example), which yields an exact sequence

$$0 \to \Omega_q \to \Omega^\text{PL}_q \to \Omega^\text{PL}_q / \Omega_q \to 0$$

in each dimension $q$, and Williamson showed in [5] that each $\Omega^\text{PL}_q / \Omega_q$ is finite. Hence $\Omega^*_e \otimes R \to \Omega^\text{PL}_e \otimes R$ is an isomorphism, so that each class in $\Omega^\text{PL}_e \otimes R$ contains at least one smooth manifold; but we already know from Lemma 1 that $L(M) = I(M)$ for smooth manifolds $M$.

Now for any PL manifold $M$ let $P(M) \in H^*(M; \mathbb{Q})$ be the image under $H^*(M; \mathbb{Q}) \to H^*(M; \mathbb{R})$ of the rational Pontrjagin class constructed by Thom in [4]. (See [1] for an alternate version of Thom's construction.) Thom's construction established the existence of unique rational classes satisfying certain axioms, which we translate as follows into real cohomology:

For each oriented PL manifold $M$ of dimension $m$ the class $P(M)$ is of the form $1 + P_1(M) + \cdots + P_{m/2}(M)$ with $P_i(M) \in H^{4i}(M; \mathbb{R})$;

for each embedding $i : N \to M$ of one oriented PL manifold into another one can assign a "normal" class $Q(N) \in H^{4*}(N; \mathbb{R})$ satisfying $P(N) \cup Q(N) = i^*P(M)$;

if $l'(M) \in H^{4*}(M; \mathbb{R})$ is the Hirzebruch polynomial constructed from $z^{1/2} / (\tanh z^{1/2})$ and $P(M)$ then the $L$-genus defined for any $4n$-dimensional compact oriented PL manifold $M$ by $L(M) = \langle l'_n(M), [M] \rangle$ satisfies $L(M) = I(M)$.

Here is the main result of this note, which permits one to conclude
that the Pontrjagin classes $p([F])$ form an extension of the Thom-Pontrjagin construction to a reasonable fibered category over the category of PL manifolds:

**PROPOSITION.** $p([\mathcal{S}(M)]) = P(M) \in H^*(M; \mathbb{R})$ for any PL manifold $M$.

**PROOF.** It suffices to verify that the classes $p([\mathcal{S}(M)])$ satisfy Thom's axioms. But $(T_1)$ is an immediate consequence of $(P_1)$, $(T_2)$ follows easily from $(P_2)$ and $(P_3)$, and $(T_3)$ follows from Lemma 2.

**REFERENCES**


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