INVARIANT SUBSPACES OF $L^\infty$ AND $H^\infty$

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Let $T$ be the unit circle, and let $L^\infty$ and $H^\infty$ be the usual spaces of bounded functions. Let $R$ be the group of rotations $z \mapsto e^{i\lambda}z$ and let $M$ be the Möbius group

$$z \mapsto \frac{e^{i\lambda}z - z_0}{1 - \overline{z}_0 z}.$$ 

Let $R$ and $M$ act on $L^\infty$ by substitution.

**Theorem 1.** Let $F$ be a closed $M$-invariant subspace of $L^\infty$, with $z \in F$ and $zF \subseteq F$. Then $F$ does not properly contain any closed $M$-invariant subspaces of finite codimension.

Examples of such subspaces $F$ are $F = L^\infty$, $F = H^\infty$, $F = A$, $F = C(T)$, $F = \mathcal{A}$, and $F = \mathcal{A}_h$. Here, $A$ is the disc algebra, $\mathcal{A}$ is the space of functions in $H^\infty$ that have radial limits along every radius, and $\mathcal{A}_h$ is the space of functions in $H^\infty$ for which the radial limit fails to exist at most on a set of Hausdorff $h$-measure 0.

**Theorem 2.** There exists an $R$-invariant closed hyperplane in $L^\infty$ that contains the space $C(T)$ but does not contain $H^\infty$.

**Corollary.** There exists an $R$-invariant closed hyperplane in $H^\infty$ that contains $A$.

**Theorem 3.** Let $B$ be a closed $R$-invariant subspace of $H^\infty$ with $B \supseteq \mathcal{A}_h$ such that either

(i) $B/\mathcal{A}_h$ is separable or

(ii) $B$ is a countably generated $\mathcal{A}_h$ module.

Then $B = \mathcal{A}_h$.

The proofs, especially of Theorem 1, are long, and we will give the details in a subsequent paper, giving here only an outline of the main steps in the proof of Theorem 1.

To begin with, we remark that $M$ is isomorphic to $\text{PSL}(2, \mathbb{R})$, which is

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SL(2, R) modulo its center. We suppose that $F \supseteq E$, where $E$ is closed and $M$-invariant and $\dim F/E < \infty$.

**Lemma 1.** A bounded, finite dimensional representation of SL(2, R) (in the discrete topology) must be trivial.

**Lemma 2.** If $F \neq E$ then $F$ contains a closed $M$-invariant subspace $E'$ such that $\dim F/E' = 1$.

From now on, we will suppose that $\dim F/E \leq 1$, and conclude that $F = E$.

**Lemma 3.** For any $f \in F$ and $\mu \in M$, $f - f \circ \mu \in E$.

**Lemma 4.** $E \supseteq A$.

**Definition.** The function $f \in L^\infty$ is $M$-analyzable, and we write $f \in \mathcal{H}_M$, if there is a complex constant in the norm-closed convex hull of the orbit of $f$ under $M$.

**Lemma 5.** $E \supseteq \mathcal{H}_M \cap F$.

**Lemma 6.** If $f$ is continuous at one point $z_0 \in T$, then $f \in \mathcal{H}_M$.

**Lemma 7** (trivial). Any $f \in L^\infty$ may be written $f = f_1 + f_2$ where $f_1$ is continuous at $+1$ and $f_2$ is continuous at $-1$.

The combination of Lemmas 3–7 implies that $F = E$.

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