THE REPRESENTATION OF LATTICES BY MODULES

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1. A quasivariety characterization of lattices representable by $\Lambda$-modules.

If $\Lambda$ is a nontrivial ring with 1, a lattice $L$ is "representable by $\Lambda$-modules" if it can be embedded in the lattice of submodules of some unitary left $\Lambda$-module $M$. This lattice of submodules is denoted $\Gamma(M;\Lambda)$.

A (lattice) "Horn formula" is an open formula:

\[(e_1 = e_2 \& e_3 = e_4 \& \ldots \& e_{n-3} = e_{n-2}) \Rightarrow e_{n-1} = e_n,\]

where $e_1, e_2, \ldots, e_n$ are lattice polynomials.

**Main Theorem.** *For every commutative ring $\Lambda$, there exists a set $J(\Lambda)$ of Horn formulas such that a lattice $L$ is representable by $\Lambda$-modules if and only if every formula of $J(\Lambda)$ is satisfied in $L$. Each member of $J(\Lambda)$ is constructible by a finite sequence of four basic operations.*

That is, the class $\mathcal{L}(\Lambda)$ of lattices representable by $\Lambda$-modules is the "quasivariety" of lattices satisfying $J(\Lambda)$, for commutative $\Lambda$.

**Outline of Proof.** For $\Lambda$ commutative, let $i: L \to \Gamma(M;\Lambda)$ be an embedding for some $M$. Without loss of generality, assume that $L$ has a smallest element $\omega$, and $i(\omega) = 0$. Motivated by the "abelian" lattice $\Gamma_f(G^N)$ of [2, 4.2] with $G = M$, we consider "constraint systems" in variables $a_k$ (corresponding to coordinate positions in $M^N$) and "auxiliary" variables $b_k$ (with existential quantifiers understood) for $k$ in $N = \{1, 2, 3, \ldots\}$. Consider $r = (d_1, d_2, d_3, d_4)$ below.

\[
\begin{align*}
(d_1) &\quad a_1 \in x_1, \quad a_2 \in x_2, \quad a_k \in \omega \quad \text{for} \quad k \geq 3 \ (x_1, x_2 \in L). \\
(d_2) &\quad b_1 \in x_3, \quad b_2 \in x_1, \quad b_k \in \omega \quad \text{for} \quad k \geq 3 \ (x_3 \in L). \\
(d_3) &\quad a_1 - a_2 - b_1 = 0. \\
(d_4) &\quad a_1 - \lambda \lambda_0 b_2 = 0 \quad (\lambda_0 \in \Lambda). 
\end{align*}
\]

A "solution" $f: N \to M$ of $r$ satisfies

\[
\begin{align*}
(e_1) &\quad f(1) \in i(x_1), \quad f(2) \in i(x_2), \quad f(k) \in i(\omega) = 0 \quad \text{for} \quad k \geq 3 \ (d_1). \\
(e_2) &\quad f(1) - f(2) \in i(x_3) \quad (d_3, b_1 \in x_3). \\
(e_3) &\quad \text{There exists } v \in i(x_1) \text{ such that } \lambda \lambda_0 v = f(1) \ (d_4, b_2 \in x_1). 
\end{align*}
\]


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From the document, the text can be transcribed as follows:

Formally, let \( N_1 = \{a_k : k \in N\} \), let \( N_2 = N_1 \cup \{b_k : k \in N\} \), and let \( M_1^{\infty} \) and \( M_2^{\infty} \) be the \( \Lambda \)-modules of all functions \( N_1 \to M \) and \( N_2 \to M \), respectively. Let a "\( \Lambda \)-equation" be a function \( g : N_2 \to \Lambda \) such that \( g(a_k) = g(b_k) = 0 \) except for finitely many \( k \) in \( N \); \( g \) determines the "linear solution set" \( g^* \) in \( \Gamma(M_2^{\infty}; \Lambda) \):

\[
g^* = \left\{ m \in M_2^{\infty} : \sum_{k=1}^{\infty} (g(a_k)m(a_k) + g(b_k)m(b_k)) = 0 \right\}.
\]

A "constraint function" is a function \( \alpha : N_2 \to L \) such that \( \alpha(a_k) = \alpha(b_k) = \omega \) except for finitely many \( k \); it determines a "box" \( i_\alpha(\alpha) \) in \( \Gamma(M_2^{\infty}; \Lambda) \):

\[
i_\alpha(\alpha) = \{ m \in M_2^{\infty} : m(c_k) \in \alpha(c_k) \quad \text{for} \quad c_k \in N_2 \}.
\]

If \( \alpha \) is a constraint function and \( G = \{g_1, g_2, \ldots, g_s\} \) is a finite (possibly empty) set of \( \Lambda \)-equations, the pair \( (G, \alpha) \) is a "constraint system". An "extended solution" \( m : N_2 \to M \) of \( (G, \alpha) \) is a member of

\[
\mu_0(G, \alpha) = i_\alpha(\alpha) \cap g_1^* \cap g_2^* \cap \cdots \cap g_s^* \quad \text{in} \quad \Gamma(M_2^{\infty}; \Lambda).
\]

A "solution" \( m' : N_1 \to M \) of \( (G, \alpha) \) is a restriction \( m' = m|N_1 \) of an extended solution \( m \). Let \( D(L; \Lambda) \) denote the set of all constraint systems. Given \( M \) and \( \iota \), define \( \mu : D(L; \Lambda) \to \Gamma(M_1^{\infty}; \Lambda) \) by the "solution set" \( \mu(G, \alpha) = \{ m|N_1 : m \in \mu_0(G, \alpha) \} \). Since \( \iota(\omega) = 0 \), \( \mu(G, \alpha) \) has "finite support" as in [2, p. 181].

Now, \( D(L; \Lambda) \) can be defined for any lattice \( L \), not just those in \( \mathcal{L}(\Lambda) \). Meet and join operations, corresponding to solution set intersection and sum, can be defined abstractly in \( D(L; \Lambda) \). We can also define "equivalence" of constraint systems, obtaining a congruence \( E(L; \Lambda) \) on \( D(L; \Lambda) \). If an embedding \( i : L \to \Gamma(M_1^{\infty}; \Lambda) \) with \( i(\omega) = 0 \) exists, the corresponding \( \mu : D(L; \Lambda) \to \Gamma(M_1^{\infty}; \Lambda) \) preserves meet and join and takes equivalent constraint systems modulo \( E(L; \Lambda) \) into the same solution set. Seven "rules of equivalence" generate \( E(L; \Lambda) \); we reconsider \( r = (d_1, d_2, d_3, d_4) \) to suggest them:

- **Constraint decrease:** The lattice constraint of \( a_1 \) can be changed to \( x_1 \land (x_2 \lor x_3) \), since \( d_3 \) can be solved for \( a_1, a_1 = a_2 + b_1 \), and \( a_2 + b_1 \) is in \( x_2 \lor x_3 \). **Linear combination augmentation:** Any \( \Lambda \)-equation of the form \( \lambda d_3 + \lambda' d_4 \) can be added to \( r \). **Defined variable augmentation:** We can "define" an unused auxiliary variable, say \( b_4 \), by adding a \( \Lambda \)-equation, say \( \lambda a_2 + \lambda' b_2 + b_4 = 0 \), if we change the lattice constraint of \( b_4 \) to \( x_2 \lor x_1 \) \((-\lambda a_2 - \lambda' b_2 \) is in \( x_2 \lor x_1 \)). **Union augmentation:** Add the \( \Lambda \)-equation \( a_2 - b_7 - b_9 = 0 \), for example, expressing a variable \( a_2 \) as a sum of two unused auxiliary variables. Then change the lattice constraints of \( b_7 \) and \( b_9 \) to some \( x_4 \) and \( x_5 \) in \( L \), respectively, such that \( x_2 \subset x_4 \lor x_5 (a_2 \subset x_2) \). **Null variable augmentation:** Terms \( \lambda a_5 \) and \( \lambda' a_5 \)
can be added to the $\Lambda$-equations $d_3$ and $d_4$, respectively, since $a_5 \in \omega$ and $\iota(\omega) = 0$. **Inessential variables augmentation:** We can add finitely many $\Lambda$-equations in the variables $b_k$, $k \geq 3$, and make finitely many arbitrary changes in the lattice constraints of those variables. **Renumbering:** $b_1$ and $b_2$ can be replaced throughout $r$ by any two other auxiliary variables.

The solution set of $r$ is unchanged by any of the above modifications. The primary fact about $M(L; \Lambda) = D(L; \Lambda)/E(L; \Lambda)$ is that it is an abelian lattice under the induced meet and join. Intuitively, $M(L; \Lambda)$ acts like the lattice of submodules with finite support of $M^N$, for some hypothetical $\Lambda$-module $M$.

Associated with any abelian lattice $X$ is a small abelian category $A_X$ [2, Main Theorem]. We next construct for each object $A$ of $A_{M(L; \Lambda)}$ a ring homomorphism $\zeta_A$ preserving 1 from $\Lambda$ into the ring of endomorphisms of $A$ ($\zeta_A(\lambda)$ is a formal analogue of $\lambda 1_A$). Let $\text{Ab}$ and $\Lambda$-$\text{Mod}$ be the usual categories of abelian groups and of $\Lambda$-modules, respectively. By [1, Theorem 7.14], there exists an exact embedding functor $F: A_{M(L; \Lambda)} \to \text{Ab}$. Defining $\lambda x = (F(\zeta_A(\lambda)))(x)$ makes $F(A)$ into a $\Lambda$-module, denoted $G(A)$ ($F(\zeta_A(\lambda)) = \lambda 1_{G(A)}$). We can prove that $\zeta_B(\lambda)f = f\zeta_A(\lambda)$ for $f: A \to B$ in $A_{M(L; \Lambda)}$, so $Ff: G(A) \to G(B)$ is $\Lambda$-linear. But then $G(\Lambda)$ and $Gf = Ff$ define an exact embedding functor $G: A_{M(L; \Lambda)} \to \Lambda$-$\text{Mod}$. Because of $G$, the lattice of subobjects of each object of $A_{M(L; \Lambda)}$ is in $\mathcal{L}(\Lambda)$. But then every interval sublattice of $M(L; \Lambda)$ is in $\mathcal{L}(\Lambda)$ by [2, 3.24], and $M(L; \Lambda) \in \mathcal{L}(\Lambda)$ follows, using a direct limit of $\Lambda$-modules.

We now define a lattice homomorphism $\psi: L \to M(L; \Lambda)$, similar to $\psi$ in [2, 4.3]. For $x$ in $L$, $\psi(x)$ is the equivalence class in $M(L; \Lambda)$ of $(\varnothing, x)$ in $D(L; \Lambda)$ given by $\theta_x(a_1) = x$, $\theta_x(c_k) = \omega$ for $c_k \in N_2 - \{a_1\}$. If $\psi$ is one-to-one, it embeds $L$ into $M(L; \Lambda)$, and so $L$ is in $\mathcal{L}(\Lambda)$. Suppose $L$ is in $\mathcal{L}(\Lambda)$ with embedding $i: L \to \Gamma(M; \Lambda)$, $i(\omega) = 0$. Since equivalent constraint systems have equal solution sets, $\mu: D(L; \Lambda) \to \Gamma(M_1^\infty; \Lambda)$ induces a function $\bar{\mu}: M(L; \Lambda) \to \Gamma(M_1^\infty; \Lambda)$. Clearly $\bar{\mu}\psi(x) = \mu(\varnothing, \theta_x) = \bar{\psi}(x)$, where $\bar{\psi}: \Gamma(M; \Lambda) \to \Gamma(M_1^\infty; \Lambda)$ is given by

$$\bar{\psi}(M') = \{m \in M_1^{\infty}: m(a_1) \in M', m(a_k) = 0 \text{ for } k > 1\}.$$  

So, $\bar{\psi} = \bar{\mu}\psi$. Since $\bar{\psi}$ is one-to-one, so is $\psi$. Therefore, $L$ is in $\mathcal{L}(\Lambda)$ if and only if $\psi$ is one-to-one.

Four of the rules generating $E(L; \Lambda)$ are called “direct reductions”, namely constraint decrease, linear combination augmentation, defined variable augmentation and union augmentation. A key argument shows that $\psi$ is one-to-one iff, for each $x$ in $L$ and sequence $r_1, r_2, \ldots, r_n$ in $D(L; \Lambda)$ such that $r_1 = (\varnothing, \theta_x)$, $r_n = (G, x)$ and $r_{i+1}$ is obtained by a direct reduction of $r_i$ ($1 \leq i < n$), we have $x(a_1) = x$. Each of the infinitely
many Horn formulas of $J(\Lambda)$ is generated by a finite sequence of four operations. These operations imitate the four rules of direct reduction, with lattice polynomials replacing elements of $L$. Using the above, we show that $\psi$ is one-to-one iff every formula of $J(\Lambda)$ is satisfied in $L$, and the main theorem follows.

**Corollary.** Every abelian lattice is representable by abelian groups.

2. Comparison of classes of representable lattices. Let $\Lambda$ and $\Lambda'$ be rings with 1, not necessarily commutative. Then $\mathcal{L}(\Lambda) \subseteq \mathcal{L}(\Lambda')$ if there exists a ring homomorphism $\Lambda \rightarrow \Lambda'$ preserving 1, or if there exists a $(\Lambda', \Lambda)$-bimodule $M$ which is faithfully flat as a right $\Lambda$-module. A simple change of rings argument proves the first result. For the other: the exact embedding functor $M \otimes_{\Lambda} -$ from $\Lambda$-$\text{Mod}$ into $\Lambda'$-$\text{Mod}$ induces an embedding from the lattice of subobjects of any $M_0$ in $\Lambda$-$\text{Mod}$ into the lattice of subobjects of $M \otimes_{\Lambda} M_0$ in $\Lambda'$-$\text{Mod}$. Then $\mathcal{L}(\Lambda) = \mathcal{L}(\Lambda')$ if $\Lambda$ is a regular ring and unitary subring of $\Lambda'$, by known ring theory. Let $\mathbb{Q}$ denote the field of rationals and $\mathbb{Z}_n$ the ring of integers modulo $n, n \geq 2$. So, $\mathcal{L}(\Lambda) = \mathcal{L}(\mathbb{Q})$ if $\Lambda$ has a unitary subring isomorphic to $\mathbb{Q}$. Also, $\mathcal{L}(\Lambda) = \mathcal{L}(\mathbb{Z}_n)$ if $\Lambda$ has characteristic $n$ for $n$ a square-free number (prime, or a product of distinct primes). Let $P_\Lambda$ be the set of primes $p$ such that $1 + 1 + \cdots + 1$ ($p$ times) is invertible in $\Lambda$. If $P$ is a set of primes, let $\mathbb{Q}(P)$ be the unitary subring of $\mathbb{Q}$ generated by $\{p^{-1} : p \in P\}$. If $\Lambda$ has characteristic zero, $a$ is the two-sided ideal of torsion elements of $\Lambda$ and $P_{\Lambda/a} = P_\Lambda$, then $\mathcal{L}(\Lambda) = \mathcal{L}(\mathbb{Q}(P_\Lambda))$. So, $\mathcal{L}(\Lambda) = \mathcal{L}(\mathbb{Q}(P_\Lambda))$ if $\Lambda$ is torsion-free.

Some of the above results are the best possible. Under various hypotheses, $\mathcal{L}(\Lambda) = \mathcal{L}(\Lambda') \neq \emptyset$ is proved by constructing a Horn formula satisfied in all lattices in $\mathcal{L}(\Lambda')$ but not in all lattices in $\mathcal{L}(\Lambda)$. These formulas reflect properties of the (additive) multiples $k \cdot 1_M = 1_M + 1_M + \cdots + 1_M$ for $M$ an arbitrary $\Lambda$-module. For example, $k \cdot 1_M = 0$ if the characteristic of $\Lambda$ divides $k$, and $k \cdot 1_M$ is an automorphism if $k \cdot 1$ is invertible in $\Lambda$. So, we can show that $\mathcal{L}(\Lambda) = \mathcal{L}(\Lambda') \neq \emptyset$ if the characteristic of $\Lambda$ does not divide the (nonzero) characteristic of $\Lambda'$, and therefore $\mathcal{L}(\Lambda) \neq \mathcal{L}(\Lambda')$ if $\Lambda$ and $\Lambda'$ have different characteristics. If $p$ is a prime invertible in $\Lambda'$ but not in $\Lambda$, then $\mathcal{L}(\Lambda) - \mathcal{L}(\Lambda') \neq \emptyset$, and so $\mathcal{L}(\Lambda) \neq \mathcal{L}(\Lambda')$ if $P_\Lambda \neq P_{\Lambda'}$. If $n$ is not square-free, then there exists $\Lambda$ with characteristic $n$ such that $\mathcal{L}(\Lambda) \neq \mathcal{L}(\mathbb{Z}_n)$. Also, if $\Lambda$ has characteristic zero and torsion ideal $a$, then $\mathcal{L}(\Lambda) \neq \mathcal{L}(\mathbb{Q}(P_\Lambda))$ if $P_{\Lambda/a} \neq P_\Lambda$. If $P$ is a proper subset of the primes or is empty, then $\Lambda$ with characteristic zero exists such that $P_\Lambda = P$ but $\mathcal{L}(\Lambda) \neq \mathcal{L}(\mathbb{Q}(P))$.

The detailed proofs of these results have been submitted for publication.

C. Herrmann and W. Poguntke have recently communicated to the author a theorem which implies that $\mathcal{L}(\Lambda)$ admits ultraproducts, for
any ring $\Lambda$ with 1. It then follows nonconstructively that $\mathcal{L}(\Lambda)$ is always a quasivariety, using the known result that a class of algebras admitting isomorphic images, subalgebras, products and ultraproducts is a quasivariety. Another of their results implies that $\mathcal{L}(\Lambda)$ is not finitely first-order axiomatizable if $\Lambda$ is a unitary subring of $\mathbb{Q}$.

REFERENCES
