ANALYTICAL CIRCLE GROUP ACTIONS ON COMPACT COMPLEX MANIFOLDS

BY SHAW MONG
Communicated by Glen E. Bredon, July 12, 1972

1. Introduction. Let $M$ be a compact complex manifold (of $m$ complex dimensions), and let $G$ be a compact Lie group acting analytically on $M$. Then the Dolbeault complexes

$$0 \to \Gamma \left( \bigwedge^p (M) \right) \to \cdots \to \Gamma \left( \bigwedge^m (M) \right) \to 0,$$

$p = 0, \ldots, m$, are $G$-elliptic complexes (for the definitions and following notions see [1], [2], [3]) and their analytical indices $\chi(A^p, G)$ (or simply $\chi^p$) are elements in the group representation ring $R(G)$. Following Hirzebruch [4], we have the $\chi_y(A^p, G)$ (or $\chi_y$)-characteristic, $\sum_{p=0}^m \chi^p(-y)^p$ (here we take the alternating sum rather than the sum in [4]), which is an element in $R(G)[y]$.

Let $\mathcal{C}_k$ be the category of $(M, G)$ such that $M$ has $k$ fixed points under the analytical action of $G$, and let $\mathcal{C} = \bigcup_{k=0}^{\infty} \mathcal{C}_k$. In this note we study the category $\mathcal{C}_k$, $k = 2, 3$, for the case $G = S^1$. (Note: (i) $\mathcal{C}_1 = \emptyset$ and (ii) $\chi_y = 0$ for $(M, S^1) \in \mathcal{C}_0$.) Precisely the problem is: what are the necessary conditions for $(M, S^1) \in \mathcal{C}_k$, $k = 2, 3$, and if they do exist, what is their $\chi_y$ and the representations of $S^1$ on the tangent planes over the fixed point set? The main tools for this study are the $S^1$-index theory and Atiyah-Bott fixed point formula. Only the statement of the result is given here. The details of the proof will appear elsewhere.

2. Main theorems.

THEOREM 1. If $(M, S^1) \in \mathcal{C}$, then $\chi_y \in \mathbb{Z}[y]$. Furthermore, if at a fixed point $A$, the representation of $S^1$ on the tangent plane $T_A M$ is given by $T_A M(t) = t^{a_1} + \ldots + t^{a_m}$, where $t \in R(S^1) = \mathbb{Z}[t, t^{-1}]$, then

$$(*) \quad \chi_y = \sum_{S^1(A) = A} \prod_{i=1}^m \left( \frac{1 - yt^{a_i}}{1 - t^{a_i}} \right).$$

THEOREM 2. If $(M, S^1) \in \mathcal{C}_2$, then either (i) $M = S^2$ or (ii) (complex) dim $M = 3$. 

Key words and phrases. $G$-index, Atiyah-Bott fixed point formula, $\chi_y$-characteristic, analytic actions, representation rings.

1 This research was supported by NSF grant GU3171.

Copyright © American Mathematical Society 1973
Corollary. Let \((M, S^1) \in \mathcal{C}_2\).

(i) If \(M = S^2\), then \((\ast)\) is given by

\[
1 + y = \frac{1 - yt^a}{1 - t^a} + \frac{1 - yt^{-a}}{1 - t^{-a}}
\]

where \(a\) is a nonzero integer.

(ii) If (complex) \(\dim M = 3\), then \((\ast)\) is given by

\[
y + y^2 = \left(\frac{1 - yt^{-a-b}}{1 - t^{-a-b}}\right)\left(\frac{1 - yt^a}{1 - t^a}\right)\left(\frac{1 - yt^b}{1 - t^b}\right)
+ \left(\frac{1 - yt^{a+b}}{1 - t^{a+b}}\right)\left(\frac{1 - yt^{-a}}{1 - t^{-a}}\right)\left(\frac{1 - yt^{-b}}{1 - t^{-b}}\right)
\]

where \((a, b)\) is a pair of positive integers.

A simple example for case (i) is given by \(S^1\) acting on \(S^2\) as a rotation along the axis through north and south poles on \(S^2\). It is an interesting problem arising from (ii) that if given any pair \((a, b)\) of positive integers, can we find an analytical \(S^1\)-action of case (ii) type?

Theorem 3. If \((M, S^1) \in \mathcal{C}_3\), then \(M\) must be a complex surface and \((\ast)\) is given by

\[
1 + y + y^2 = \left(\frac{1 - yt^a}{1 - t^a}\right)\left(\frac{1 - yt^b}{1 - t^b}\right)
+ \left(\frac{1 - yt^{a-b}}{1 - t^{a-b}}\right)\left(\frac{1 - yt^{-b}}{1 - t^{-b}}\right)
+ \left(\frac{1 - yt^{b-a}}{1 - t^{b-a}}\right)\left(\frac{1 - yt^{-a}}{1 - t^{-a}}\right)
\]

where \(a \neq b\) are any nonzero integers.

Example. The linear action of \(S^1\) on \(CP(2)\) given by: \((z_0, z_1, z_2) \rightarrow (z_0, t^a z_1, t^b z_2), a \neq b,\) belongs to \(\mathcal{C}_3\).

References


Department of Mathematics, State University of New York at Albany, Albany, New York 12203
(Current address): Departement de Mathématiques, Université de Montréal, Montréal 101, Québec, Canada