Let $X$ be a $C^\infty$ vector field on the $C^\infty$ manifold $M$. We use $t \to X_t(m)$ to denote the integral curve of $X$ which passes through $m$ when $t = 0$. Let $D$ be a set of $C^\infty$ vector fields on $M$. Two points $m$ and $m'$ of $M$ are said to be $D$-connected if there exist elements $X^1, \ldots, X^k$ of $D$ and real numbers $t_1, \ldots, t_k$ such that

$$m' = X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m)\cdots)).$$

This defines an equivalence relation on $M$. The equivalence classes are called the orbits of $D$.

Let $S$ be an orbit of $D$. For each $m \in S$ and each finite sequence $\zeta = (X^1, \ldots, X^k)$ of elements of $D$, let $F_{\zeta,m}$ denote the map

$$(t_1, \ldots, t_k) \to X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m)\cdots)).$$

It is clear that $F_{\zeta,m}$ is a $C^\infty$ mapping from an open subset $U$ of $\mathbb{R}^k$ into $M$. Moreover the range of $F_{\zeta,m}$ is a subset of $S$. We topologize $S$ by the strongest topology for which all the maps $F_{\zeta,m}$ are continuous.

**Theorem 1.** $S$ is a connected $C^\infty$ submanifold of $M$.

A distribution on $M$ is a mapping $H$ which assigns to every $m \in M$ a linear subspace $H(m)$ of the tangent space of $M$ at $m$. It is not required that the dimension of $H(m)$ be constant. A vector field $X$ defined in an open subset $U$ of $M$ belongs to the distribution $H$ if $X(m) \in H(m)$ for every $m \in U$. We say that $H$ is a $C^\infty$ distribution if, for every $m \in M$ and every $v \in H(m)$, there exists a $C^\infty$ vector field $X$ such that $X$ belongs to $H$ and $X(m) = v$. If $D$ is a set of vector fields and $H$ is a distribution, we say that $H$ is $D$-invariant if, whenever $m \in M$, $X \in D$, and $t$ is a real number such that $X_t(m)$ is defined, it follows that the differential of $X_t$ maps $H(m)$ into $H(X_t(m))$. Given a set $D$ of $C^\infty$ vector fields on $M$, there exists a smallest distribution $H$ which is $D$-invariant and is such that every element of $D$ belongs to $H$. Let this distribution be denoted by $P_D$. Then $P_D$ is a $C^\infty$ distribution.

Integral manifolds and maximal integral manifolds of a distribution


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are defined in the usual way. In the statement of the following theorem, the orbits of $D$ are given the topology that was defined in the remarks preceding Theorem 1, and the differentiable structure whose existence is asserted in Theorem 1. This structure is clearly unique.

**Theorem 2.** The orbits of $D$ are maximal integral manifolds of $P_D$.

The preceding result shows that the distribution $P_D$ is integrable, in the sense that through every point of $M$ there passes a maximal integral manifold of $P_D$. It turns out that the converse is also true: Every integrable distribution is of the form $P_D$ for some $D$.

**Theorem 3.** Let $H$ be a $C^\infty$ distribution and let $D$ be a set of $C^\infty$ vector fields such that, for every $m \in M$, $H(m)$ is the linear hull of the vectors $X(m)$, $X \in D$. Then the following conditions are equivalent:

(a) Through every point of $M$ there passes an integral manifold of $H$.

(b) Through every point of $M$ there passes a maximal integral manifold of $H$.

(c) $H = P_D$.

(d) For every $m \in M$ there exist elements $X^1, \ldots, X^k$ of $D$ such that $H(m)$ is the linear hull of $X^1(m), \ldots, X^k(m)$ and that the following holds:

For every $X \in D$ there exist $\varepsilon > 0$ and $C^\infty$ functions $f^j_i: (-\varepsilon, \varepsilon) \to \mathbb{R}$ ($1 \leq i, j \leq k$) such that, for $-\varepsilon < t < \varepsilon$,

$$[X, X^i](X_t(m)) = \sum_{i=1}^{k} f^j_i(t)X^j(X_t(m)).$$

Proofs of our three theorems will appear elsewhere. We now discuss the connection of our results with those of Chow [1], Hermann [2], Lobry [3], Nagano [5] and Matsuda [4]. Let $D^*$ be the smallest set of vector fields which contains all the elements of $D$ and which is closed under Lie brackets. Let $I_D(m)$ denote the linear hull of the vectors $X(m)$, $X \in D^*$. Chow proved that the orbits of $D$ are precisely the connected components of $M$, provided $I_D(m)$ has maximal dimension for each $m$. If the assumption of Chow's theorem is satisfied, it is clear that $I_D(m) = P_D(m) =$ tangent space of $M$ at $m$. Therefore Chow's result is a particular case of Theorem 2. Hermann and Lobry studied (under the name of "leafs") the orbits of $D$ under the assumption that the distribution $I_D$ is integrable. By applying Chow's theorem to the integral manifolds of $I_D$, they concluded that the orbits of $D$ are the maximal integral manifolds of $I_D$. Theorem 1 shows that the orbits of $D$ are always smooth submanifolds of $M$, and that the integrability of $I_D$ is not needed. Theorem 2 shows that the orbits are the maximal integral manifolds of an integrable distribution $P_D$. In general,
$P_D$ will not coincide with $I_D$ and, therefore, $I_D$ is not the “correct” distribution to look at.

The integrability of smooth distributions $H$ which are involutive \( (i.e., \text{whenever } X \text{ and } Y \text{ belong to } H \text{ then } [X, Y] \text{ belongs to } H) \) is a classical result if the dimension of $H(m)$ is constant. In general \( (i.e., \text{when } H \text{ has “singularities”}) \) the condition that $H$ be involutive is necessary but not sufficient. Sufficient conditions were given by Hermann [2] \( (H \text{ is “locally finitely generated”)}, Lobry [3] \( (H \text{ is “locally of finite type”}) \) and Matsuda [4] \( (H \text{ satisfies a “convergence condition”)}. \) It is easy to see that Lobry’s condition \( (\text{which is weaker than Hermann’s}) \) implies condition \( (d) \) of Theorem 3. Also, Matsuda’s condition implies \( (c) \). Therefore, these results are all contained in Theorem 3.

Nagano [5] proved the result \( (\text{stated by Hermann in [2]}) \) that every analytic involutive distribution on a real analytic manifold $M$ is integrable. This also follows from Theorem 3, because in the analytic case condition \( (d) \) is always satisfied \( (cf. \text{Lobry [3]}). \)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903 (Current address)