Periodic and Homogeneous States on a Von Neumann Algebra. I

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This paper is devoted to announcing a structure theorem for von Neumann algebras admitting a periodic homogeneous faithful state (see Definitions 1 and 2).

Let \( \mathcal{M} \) be a von Neumann algebra. Suppose that \( \phi \) is a faithful normal state on \( \mathcal{M} \). We denote by \( \sigma_\phi^t \) the modular automorphism group of \( \mathcal{M} \) associated with \( \phi \). Let \( G(\phi) \) denote the group of all automorphisms of \( \mathcal{M} \) which leave \( \phi \) invariant. We introduce the following terminologies concerning \( \phi \).

**Definition 1.** If there exists \( T > 0 \) such that \( \sigma_\phi^T \) is the identity automorphism of \( \mathcal{M} \), denoted by \( 1 \), then we call \( \phi \) periodic. The smallest such number \( T \) is called the period of \( \phi \).

**Definition 2.** We call \( \phi \) homogeneous if \( G(\phi) \) acts ergodically on \( \mathcal{M} \); that is, the fixed points of \( G(\phi) \) are only scalar multiples of the identity.

**Definition 3.** We call \( \phi \) ergodic if \( \{ \sigma_\phi^t \} \) is ergodic.

The ergodicity of \( \phi \) implies the homogeneity of \( \phi \), since \( \{ \sigma_\phi^t \} \) is contained in \( G(\phi) \). Furthermore, if \( \mathcal{M} \) admits an ergodic state, then \( \mathcal{M} \) must be a factor.

Now, suppose \( \phi \) is a periodic homogeneous faithful normal state on \( \mathcal{M} \), which will be fixed throughout the discussion. Considering the cyclic representation of \( \mathcal{M} \) induced by \( \phi \), we assume that \( \mathcal{M} \) acts on a Hilbert space \( \mathcal{H} \) with a distinguished cyclic vector \( \xi_0 \) such that \( \phi(x) = (x|\xi_0|\xi_0) \), \( x \in \mathcal{M} \). According to the theory of modular Hilbert algebras (which the author proposes to call Tomita algebras), there exists the positive self-adjoint operator \( \Delta \) on \( \mathcal{H} \) and the unitary involution \( J \) on \( \mathcal{H} \) such that

\[
\sigma_\phi^t(x) = \Delta^{it}x\Delta^{-it}, \quad x \in \mathcal{M};
\]

\[
\Delta^{it}\xi_0 = \xi_0;
\]

\[
J\mathcal{M}J = \mathcal{M}^*; \quad J\Delta^{it}J = \Delta^{-it}.
\]

Put \( \alpha = e^{-2\pi i/T} \) with \( T \) the period of \( \phi \). Obviously, we have \( 0 < \alpha < 1 \). We introduce the following notations:

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For $n = 0, \pm 1, \pm 2, \ldots$. Then $M_0$ is nothing but the centralizer $M_0$ of $\phi$ in the sense of [11, Definition 8.6]. The ergodicity of $G(\phi)$ implies that $M_n \neq \{0\}$ for every integer $n$. The subspace $M_n$ of $M$ is also given by

$$M_n = \left\{ x \in M : \phi(x) = \alpha^n \phi(yx) \text{ for every } y \in M \right\},$$

due to Størmer [9].

**Lemma 4.** We have the following:

(i) $M_n M_m \subset M_{n+m}$, $M_n^* = M_{-n}$;

(ii) $M_n H_n \subset H_{n+m}$, $JH_n = J_{-n}$;

(iii) $H = \sum_{n = -\infty}^{\infty} H_n$;

(iv) $H_n = \left[ M_n \xi_0 \right]$.

It is easily seen that the algebraic direct sum $\sum_{n = -\infty}^{\infty} M_n$ is a $\sigma$-weakly dense *-subalgebra of $M$. If $N$ is a von Neumann subalgebra of $M$ invariant under $\sigma$, then the algebraic direct sum $\sum_{n = -\infty}^{\infty} (N \cap M_n)$ is also a $\sigma$-weakly dense *-subalgebra of $N$. Since $M_n^* M_n \subset M_0$ and $M_n M_n^* \subset M_0$, the absolute value $|x|$ of every element $x$ in $M_n$ falls in $M_0$. Hence, if $x \in M_n$ commutes with $M_0$, then $x$ commutes with $x^* x$ and $xx^*$, so that $x$ is normal, that is, $x^* x = xx^*$. But this is impossible unless $x$ is in $M_0$ because $\alpha^n \phi(x^* x) = \phi(xx^*)$. Thus we obtain the following:

**Proposition 5.** The relative commutant $M_0 \cap M$ of $M_0$ in $M$ is contained in $M_0$ as the center of $M_0$, denoted by $Z_0$.

We denote by $\pi_n$ the normal representation of $M_0$ on $H_n$ defined by restricting the action of $M_0$ to $H_n$. We also define the antirepresentation $\pi_n'$ of $M_0$ on $H_n$ by

$$\pi_n'(a) = J\pi_n(-a)^* J, \quad a \in M_0.$$  

For each $x \in M_n$, we have

$$\pi_n(a)x_0 = ax_0;$$

$$\pi_n'(a)x_0 = xa_0, \quad x \in M_0.$$  

Hence $\pi_n$ and $\pi_n'$ commute. Making use of the ergodicity of $\phi$, we can prove the following:

**Lemma 6.** Both $\pi_n$ and $\pi_n'$ are faithful.

For each $g \in G(\phi)$, we define a unitary operator $U(g)$ on $H$ by

$$U(g)x_0 = g(x)x_0, \quad x \in M.$$  

Then the map $:g \in G(\phi) \mapsto U(g)$ is a representation of $G(\phi)$ and covariant
with the action of $\mathcal{M}$. It is easily seen that

$$U(g)\pi_n(x)U(g)^* = \pi_n \circ g(x);$$

$$U(g)\pi'_n(x)U(g)^* = \pi_n \circ g(x), \quad x \in \mathcal{M}_0, g \in G(\phi).$$

The ergodicity of $G(\phi)$ on $\mathcal{M}_0$ yields that the coupling operator of $\{\pi_n(\mathcal{M}_0), \mathfrak{S}_n\}$ in the sense of Griffin [6] is a scalar multiple of the identity. Therefore, $\{\pi_n(\mathcal{M}_0), \mathfrak{S}_n\}$ has either a separating vector or a cyclic vector.

**Lemma 7.** For $n \geq 1$, $\{\pi_n, \mathfrak{S}_n\}$ does not have a separating vector.

**Proof.** Since every $\xi \in \mathfrak{S}_n$ is analytic for $\Delta^u$, there exists a closed operator $a$ affiliated with $\mathcal{M}$ such that $\xi = a\xi_0$. We can choose $a$ so that $\Delta^u a \Delta^{-u} = \alpha^u a$. Let $a = uh$ be the polar decomposition of $a$. Then $h$ is affiliated with $\mathcal{M}_0$ and $u \in \mathcal{M}_n$. If $\xi$ is separating, then $x\xi = 0, x \in \mathcal{M}_0$, implies $x = 0$, so that $xu = 0$ implies $x = 0$. Hence $uu^* = 1$. But $x^*\phi(uu^*) = \phi(uu^*) = 1$, so that $\phi(uu^*) = \alpha^{-n} > 1$ if $n \geq 1$, a contradiction.

Therefore, $\{\pi_n, \mathfrak{S}_n\}, n \geq 1$, has a cyclic vector $\xi$, which is separating for $\pi_{-n}(\mathcal{M}_0)$. If $a = ku$ is the right polar decomposition of the above $a$ in Lemma 7, then $ux = 0, x \in \mathcal{M}_0$, implies $x = 0$, so that we have $u^*u = 1$, and $\phi(uu^*) = \alpha^n$. We choose an element $u_1$ in $\mathcal{M}_1$ with $u_1^*u_1 = 1$, and fix it. Then $u_1^*$ falls in $\mathcal{M}_n$ for $n \geq 1$, and $\mathcal{M}_n = \mathcal{M}_0 u_1^n$ because $\mathcal{M}_n u_1^n \subseteq \mathcal{M}_0$. Therefore we have

$$\mathcal{M}_n = \mathcal{M}_0 u_1^n;$$

$$\mathcal{M}_{-n} = u_1^{*n} \mathcal{M}_0, \quad n = 1, 2, \ldots .$$

Thus the von Neumann algebra $\mathcal{M}$ is generated by $\mathcal{M}_0$ and the isometry $u_1$. The choice of $u_1$ is unique in the following sense:

**Lemma 8.** Every partial isometry $v$ in $\mathcal{M}_1$ is of the form $wu_1$ with a partial isometry $w$ in $\mathcal{M}_0$. Let $e_{-n}$ denote the projections $u_1^nu_1^*\mathcal{M}_0$ for $n \geq 1$. Then Lemma 8 implies, together with the ergodicity of $G(\phi)$, that

$$e_{-n}^\perp = \alpha^n 1.$$

Thus we conclude that $\mathcal{M}_0$ is of type $I_1$. We denote by $e_n$ the projection $J e_{-n} J$ in $\mathcal{M}_0$. Let $\mathfrak{S}_n = e_n \mathfrak{S}_0$, for every integer $n$.

Define an isomorphism $\theta$ of $\mathcal{M}_0$ onto $e_{-1} \mathcal{M}_0 e_{-1}$ by $\theta(x) = u_1 xu_1^*$, $x \in \mathcal{M}_0$. Then the isomorphism $\theta$ induces an automorphism $\tilde{\theta}$ of $\mathfrak{Z}_0$ by the equality $\theta(a) = \tilde{\theta}(a)e_{-1}, a \in \mathfrak{Z}_0$. It follows from Lemma 8 that $\tilde{\theta}$ does not depend on the choice of $u_1$.

**Proposition 9.** The center $\mathfrak{Z}$ of $\mathcal{M}$ is precisely the fixed point subalgebra of $\mathfrak{Z}_0$ with respect to $\tilde{\theta}$. Therefore, $\mathcal{M}$ is a factor if and only if $\tilde{\theta}$ is ergodic on $\mathfrak{Z}_0$. 
PROPOSITION 10. For $n \geq 1$, we have
\[
\{\pi_n, S_n\} \cong \{\pi_0, S_n\} \quad \text{and} \quad \{\pi_{-n}, S_{-n}\} \cong \{\theta^n, S_{-n}\},
\]
where $\{\pi_0, S_n\}$ means the restriction of $\pi_0$ to the invariant subspace $S_n$.

We denote by $\phi_0$ the restriction of $\phi$ to $M_0$.

THEOREM 11. In the pre-Hilbert space metric given by the state $\phi$, the von Neumann algebra $M$ is decomposed as
\[
M = \cdots \oplus u^n_1 M_0 \oplus G \cdots \oplus u^n_1 M_0 \oplus M_0u_1 \oplus \cdots \oplus M_0u^n_1 \oplus \cdots.
\]
The algebraic structure of $(M, \phi)$ is determined by $\{M_0, \theta, \phi_0\}$ in the following sense: Let $\mathcal{M}$ be another von Neumann algebra equipped with a periodic homogeneous faithful state $\tilde{\phi}$ of period $T$ and let $\mathcal{M}$ be decomposed with respect to $\tilde{\phi}$ as
\[
\mathcal{M} = \cdots \oplus \tilde{u}^n_1 \tilde{M}_0 \oplus \cdots \oplus \tilde{u}^n_1 \tilde{M}_0 \oplus \tilde{M}_0 \tilde{u}_1 \oplus \cdots \oplus \tilde{M}_0 \tilde{u}^n_1 \oplus \cdots.
\]
Suppose $\tilde{u}_1$ gives rise to an isomorphism of $\tilde{\phi}$ of $\tilde{M}_0$ onto $\mathcal{M}$ onto $e^{-1} M_0 e^{-1}$. Then there exists an isomorphism $\sigma$ of $M$ onto $\mathcal{M}$ with $\phi = \tilde{\phi} \circ \sigma$ if and only if there exists an isomorphism $\sigma_0$ of $M_0$ onto $\mathcal{M}_0$ and a partial isometry $w$ in $M_0$ such that $w\theta(x)w^* = \sigma_0^{-1} \circ \tilde{\phi} \circ \sigma_0(x)$, $x \in M_0$, and $\phi_0 = \tilde{\phi}_0 \circ \sigma$, where $\phi_0$ (resp. $\tilde{\phi}_0$) means the restriction of $\phi$ (resp. $\tilde{\phi}$) to $M_0$ (resp. $\mathcal{M}_0$).

Conversely, if $M_0$ is a von Neumann algebra of type II$_1$. Let $\epsilon$ be a projection of $M_0$ with $e^b = \alpha$, $0 < \alpha < 1$. Suppose $\theta$ is an isomorphism of $M_0$ onto $eM_0e$. Then $\theta$ induces an automorphism $\tilde{\theta}$ of the center $Z_0$ of $M_0$ such that $\theta(a)e = \theta(a)$, $a \in Z_0$. Let $\phi_0$ be a $\tilde{\theta}$-invariant faithful normal state on $Z_0$. We extend $\phi_0$ to a faithful normal trace on $M_0$ by $\phi_0(x) = \phi_0(x^b)$, $x \in M_0$. Suppose $G$ denotes the group of all automorphisms $g$ of $M_0$ such that there exists a partial isometry $w_\theta$ in $M_0$ with $g \circ \theta \circ g^{-1}(x) = w_\theta \theta(x)w_\theta^*$, and such that $\phi_0 \circ g = \phi_0$ (this is satisfied automatically if $\tilde{\theta}$ is ergodic). Such an automorphism is called admissible.

THEOREM 12. In the above situation, if $G$ acts ergodically on the center $Z_0$, then there exists a von Neumann algebra $\mathcal{M}$ with a periodic homogeneous faithful state $\phi$ of period $T = -2\pi/\log \alpha$ such that $\{M_0, \theta, \phi_0\}$ appears in the decomposition of $\mathcal{M}$ associated with $\phi$ as described in Theorem 11.

We denote by $\mathcal{R}(M_0, \theta, \phi_0)$ the von Neumann algebra determined by $(M_0, \theta, \phi_0)$ in Theorems 11 and 12. We can describe the automorphism group $G(\phi)$ in terms of $G$ and the unitary group of $Z_0$. In order to distinguish the algebraic type of $\mathcal{R}(M_0, \theta, \phi_0)$, we employ new results of A. Connes [4] concerning modular automorphism groups.
For a von Neumann algebra $\mathcal{M}$, let $\text{Aut}(\mathcal{M})$ (resp. $\text{Int}(\mathcal{M})$) denote the group of all (resp. inner) automorphisms of $\mathcal{M}$. Let $\text{Out}(\mathcal{M})$ denote the quotient group $\text{Aut}(\mathcal{M})/\text{Int}(\mathcal{M})$. A. Connes showed recently that the canonical image $\hat{\sigma}^T$ of the modular automorphism group $\sigma^T$ in $\text{Out}(\mathcal{M})$ does not depend on the choice of $\phi$; hence we denote it simply by $\hat{\sigma}^T$. Furthermore he proved that if $\sigma^T$ is inner for some $T > 0$, then $\sigma^T$ is given by a unitary operator in the center of the centralizer $\mathcal{M}_\phi$ of $\phi$.

Now, we return to the original situation. In order to avoid any possible confusion, we denote by $T_0$ the period of our state $\phi$.

**Theorem 13.** For $T > 0$, $\sigma^T$ is inner, that is, $\hat{\sigma}^T = \text{identity}$, if and only if $\alpha^{-iT}$ is a point spectrum of the automorphism $\theta$ of $\mathcal{Z}_0$.

Therefore, if we have ergodic automorphisms $\theta$ in $\mathcal{Z}_0$ of different point spectral type, then the resulting factors $\mathcal{H}(\mathcal{M}_0, \theta, \phi_0)$ are nonisomorphic.

**Examples.** Let $\mathcal{F}$ denote a hyperfinite $II_1$-factor and $\mathcal{A} = L^\infty(0,1)$. Let $\mathcal{M}_0 = \mathcal{F} \otimes \mathcal{A}$. For $0 < \alpha < 1$, we choose a projection $f \in \mathcal{F}$ with $\tau(f) = \alpha$, where $\tau$ is the canonical trace of $\mathcal{F}$. It is then known that there exists an isomorphism $\theta_\alpha$ of $\mathcal{F}$ onto $f \mathcal{F} f$. Let $\theta$ be an ergodic automorphism of $\mathcal{A}$ with invariant faithful normal state $\mu$. Let $\theta_0 = \theta_1 \otimes \theta$ and $\phi_0 = \tau \otimes \mu$. Then the triplet $\{\mathcal{M}_0, \theta, \phi_0\}$ satisfies all our requirements, since the automorphism $\text{id} \otimes \theta^n$, $n = 0 \pm 1, \pm 2, \ldots$, are admissible and ergodic on the center $\mathcal{Z}_0 = 1 \otimes \mathcal{A}$. Thus, if we choose various kinds of ergodic automorphisms $\theta$, then we get different kinds of modular groups $\hat{\sigma}$, as well as different factors.

**References**