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THEOREM. The operator $S \oplus M$ is of the form normal plus compact.

We begin by obtaining a matrix representation for $M$ on $L^2(\Delta)$. We do this in the following rather roundabout manner: Let $A^2$ denote the analytic functions in $L^2(\Delta)$. Then $A^2$ is a Hilbert space and has as an orthonormal basis $\{(n + 1)/\pi^{1/2}z^n\}_{n=0}^{\infty}$ [5, p. 15]. Clearly $A^2$ is invariant for $M$ and $B = M|A^2$ is easily seen to be unitarily equivalent to the weighted shift $S_1$ with weights $\{n/(n + 1)\}_{n=1}^{\infty}$. Since $M$ is the minimal normal extension of the subnormal operator $B$, the characterization of the minimal normal extension of a subnormal weighted shift obtained in [6] yields the following matrical description of $M$. The operator $M$ is unitarily equivalent to the matrix

$$
N = \begin{bmatrix}
S_1 & D_2 & 0 \\
S_2 & D_3 & \\
& & \ddots \\
0 & & & \ddots
\end{bmatrix}
$$

where $S_n$ is the weighted shift with weights

$$
\left\{ \begin{array}{l}
a_k^{(n)} = \left( \frac{kk}{(n + k - 1)(n + k)} \right)^{1/2} \\
\end{array} \right\}_{k=1}^{\infty}
$$

and $D_n$ is the diagonal operator with entries

$$
\left\{ \begin{array}{l}
b_k^{(n)} = \left( \frac{(n - 1)(n - 1)}{(n + k - 2)(n + k - 1)} \right)^{1/2} \\
\end{array} \right\}_{k=1}^{\infty}
$$

To prove this one need only verify the identities $(b_k^{(n+1)})^2 = (a_k^{(n)})^2 + (b_k^{(n)})^2 - (a_k^{(n)})^2$ and $a_k^{(n+1)}b_k^{(n+1)} = a_k^{(n)}b_{k+1}^{(n+1)}$ for $k, n = 1, 2, \ldots$ ($a_0^{(n)} = 0$ by definition) from [6].

If we can show that there exists a compact operator $K'$ such that $N + K'$ is unitarily equivalent to $S \oplus N$, then we will have shown that there exists a compact operator $K$ such that $M + K$ is unitarily equivalent to $S \oplus M$, thus answering the Douglas–Fillmore question. The existence of such a $K'$ follows from the following facts:

(I) There exists a compact operator $K_1$ such that $S_1 + K_1 = S$. (Let
$K_1$ be the weighted shift with weights $\{1 - [n/(n + 1)]^{1/2}\}_{n=1}^\infty$ [5, p. 86].

(II) $D_2$ is a compact operator, in fact each $D_n$ is compact [5, p. 86].

(III) $S_n - S_{n+1}$ and $D_n - D_{n+1}$ are compact for each $n = 1, 2, 3, \ldots$, and $\|S_n - S_{n+1}\|, \|D_n - D_{n+1}\| < 1/n$ for each $n = 1, 2, \ldots$ (Here $S_n - S_{n+1}$ is the weighted shift with weights $\{c_k^{(n)} = a_k^{(n)} - a_k^{(n+1)}\}_{k=1}^\infty$, and $D_n - D_{n+1}$ is the diagonal operator with entries $\{d_k^{(n)} = b_k^{(n)} - b_k^{(n+1)}\}_{k=1}^\infty$.)

Straightforward calculations reveal that $c_k^{(n)}, \|d_k^{(n)}\| < 1/n$ for all $n, k = 1, 2, \ldots$. The first fact shows that $S_n - S_{n+1}$ and $D_n - D_{n+1}$ are compact, while the second fact shows that $\|S_n - S_{n+1}\|, \|D_n - D_{n+1}\| < 1/n$.

Facts (I), (II), and (III) imply that if

$$K' = \begin{bmatrix} K_1 & -D_2 & 0 & \cdots \\ S_1 - S_2 & D_2 - D_3 & & \\ & S_2 - S_3 & D_3 - D_4 & \\ & & \vdots & \ddots \end{bmatrix}$$

then $K'$ is a compact operator, and $N + K'$ is unitarily equivalent to $S \oplus N$. Hence there exists a compact operator $K$ such that $S \oplus M$ is unitarily equivalent to $M + K$.

REMARKS. (1) Since $S \oplus M$ is normal plus compact, there exists a normal operator $N$ and a compact operator $K$ such that the residual spectrum of $N + K$ is an open set.

(2) Since the self-commutator of $S \oplus M$ is trace class with nonzero trace, it is impossible to write $S \oplus M$ as normal plus trace class.

(3) Using Berg’s theorem [1] one can easily verify that $S \oplus M$ is normal plus compact, where $S$ is a unilateral shift of finite multiplicity and $M$ is any normal operator whose spectrum contains $\Delta$.

(4) Let $R$ be a normal operator whose spectrum is $\Delta \cap \{z : \text{Re}(z) \geq 0\}$. Then $S \oplus R$ is an example of a nonquasitriangular operator whose square is normal plus compact and hence quasitriangular [see 3].

REFERENCES


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