BOUNDARY VALUES IN CHROMATIC GRAPH THEORY

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Let $G$ be a planar graph drawn in the plane so that its outer boundary $\Gamma$ is a $k$-cycle. A four-coloring of $\Gamma$ is admissible if it extends to a four-coloring of all of $G$. Let $\psi$ be the number of admissible boundary colorings, and we suppose the truth of the Four-Color Conjecture in the theorems marked with a * below.

**CONJECTURE.** $\psi \geq 3 \cdot 2^k$ ($k = 3, 4, \ldots$). (The sign of equality holds if $G$ is a triangulation of a $k$-cycle with no interior vertices.)

**THEOREM 1.** $\psi \geq 24F_{k-1} \geq C((1 + 5^{1/2})/2)^k$, where $F_k$ is the $k$th Fibonacci number.

**THEOREM 2.** $\psi \geq 3 \cdot 2^k$ for $k = 3, 4, 5, 6$.

A graph is totally reducible (t.r.) if every four-coloring of the boundary is admissible (i.e., $\psi = 3^k + (-1)^k \cdot 3$).

**THEOREM 3.** For each $k$ there is a t.r. graph $G$ whose boundary is a $k$-cycle and whose interior is a triangulation.

An annulus $G_{kl}$ is an $l$-cycle drawn interior to a $k$-cycle, with a maximum number of nonintersecting edges connecting the two cycles. The vertices of the $l$-cycle are $u_1, u_2, \ldots, u_l$, and $\rho(u)$ is the valence of the vertex $u$.

**THEOREM 4.** An annulus $G_{kl}$ is t.r. iff it has none of the following properties:
1. $\rho(u_i) \geq 6$; 2. $\rho(u_i) = \rho(u_j) = 5$ ($i \leq k - 3$) and $\rho(u_i) = 4$ for all $i$ in $1 < i < j$; 3. $\rho(u_i) = \rho(u_j) = 5$, $\rho(u_i) = 4$ for all $i$ in $1 < i < j$, $j = k - 2$, $k$ even; 4. $\rho(u_i) = 5$, $\rho(u_j) = 4$ for all $1 < i < l$, $l$ odd.

**THEOREM 5.** An annulus $G_{kl}$ satisfies the Conjecture stated above.

Proofs will appear elsewhere.

**REFERENCES**


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