1. Introduction. The irreducible modular representations of the finite Chevalley groups (and their twisted analogues) have been described by C. W. Curtis and R. Steinberg (see [1], [2], [8]). In this note we outline some parallel results on principal indecomposable modules (PIM's), for which proofs will appear elsewhere. The groups $\text{SL}(2, q)$ are already treated in detail in [5], based in part on the method of A.V. Jeyakumar [6].

$K$ denotes an algebraically closed field of prime characteristic $p$, over which all modules are assumed to be finite dimensional. Our notation resembles that of [4], [5]: $G$ is a simply connected algebraic group (of simple type), $\mathfrak{g}$ its Lie algebra, $\mathcal{U}$ the restricted universal enveloping algebra of $\mathfrak{g}$, $G_q$ the group of rational points of $G$ over a field of $q$ elements, $\mathcal{R}_q$ the group algebra of $G_q$ over $K$. (When $q = p$, we write simply $G, \mathcal{R}$.)

**Example.** $G = \text{SL}(2, K), \mathfrak{g} = \mathfrak{sl}(2, K), G_q = \text{SL}(2, q)$.

The set $\Lambda$ of restricted highest weights (determined by integers between $0$ and $p - 1$) indexes the (classes of) irreducible modules $M_{\lambda}$ for $\mathfrak{g}$ (or $\mathcal{R}$). If $\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_t p^t (\lambda_i \in \Lambda)$, then the twisted tensor product modules $M_{\lambda} = M_{\lambda_0} \otimes M_{\lambda_1}^{(p)} \otimes \cdots \otimes M_{\lambda_t}^{(p^t)}$ exhaust the (classes of) irreducible modules for $\mathcal{R}_q (q = p^{k+1})$ and for $G$ (as $k$ runs over all nonnegative integers). Denote by $U_\lambda, R_\lambda, R_\mathcal{R}$ the respective PIM of $\mathfrak{g}, R, \mathcal{R}$ having top composition factor $M_{\lambda_0}, M_{\lambda_1}, M_{\lambda}$.

The only irreducible module which is also projective is the Steinberg module $M_{\sigma} = U_\sigma = R_\sigma, \sigma = (p - 1)\delta, \delta = \text{half-sum of positive roots}$. A similar statement is true for $M_{\sigma} = R_\sigma (\sigma = \sigma + \sigma p + \cdots + \sigma p^t)$.

2. Projective modules.

**Lemma.** Let $V, W$ be modules for the restricted universal enveloping algebra of a restricted Lie algebra, with $W$ projective. Then $V \otimes W$ is also projective.

This is proved in [7]. The analogous statement for the group algebra of a finite group is well known [3, Exercise 2, p. 426].

We apply the lemma as follows. For $\mu \in \Lambda$, define $T_\mu = M_{\mu} \otimes M_{\sigma}$ ($\sigma$ as above). This is a module for $G, \mathcal{R}, \mathcal{U}$, and is projective for $\mathcal{R}, \mathcal{U}$ (since $M_{\sigma}$ is). In particular, $T_\sigma$ is the direct sum of certain $\mathcal{U}$-modules $U_\lambda$.\[\text{AMS (MOS) subject classifications (1970). Primary 20C20; Secondary 20G40, 17B10.}

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If \( \mu \in \Lambda \), define its opposite \( \mu^0 \) to be \( \tau_0(\mu + \delta) - \delta, \tau_0 \) the unique element of the Weyl group which interchanges positive and negative roots. Our main result can now be formulated.

**Theorem A.** Set \( \lambda = (\mu - \delta)^0 \). Then \( U_\lambda \) occurs precisely once as a \( \mathcal{U} \)-summand of \( T_\mu \) and is stable under \( G \), therefore is also a projective \( \mathcal{R} \)-module involving \( R_\lambda \) as a summand. In particular, \( \dim R_\lambda \leq \dim U_\lambda \).

The proof uses some ideas from [4]. For \( G = \text{SL}(2, K) \), a result of this type was first noticed empirically by the second author.

**Remarks.**

1. One can effectively compute (at least for small rank and small \( p \)) the modules \( T_\mu \), starting with the known decomposition of tensor products of irreducible modules in characteristic 0 and then reducing modulo \( p \). Using this approach and other data, the first author computed the Cartan invariants of \( \text{SL}(3, 5) \), avoiding Brauer's method.

2. From the tensor product construction (and knowledge of the modules \( M_\mu \)) one also gets an effective, but lengthy, algorithm for computing the "decomposition" numbers \( d_{\lambda\mu} \) which figure in [4]. This in turn yields the Cartan invariants of \( \mathcal{U} \).

Call \( \lambda \in \Lambda \) regular if \( \lambda = \sum m_i \lambda_i \) with all \( m_i \) nonzero (\( \lambda_i \in \Lambda \) fundamental dominant weights). Empirical evidence, along with some heuristic arguments, suggests the following conjecture, which is true in rank 1 ([5], [6]) and also for \( G = \text{SL}(3, 5), \text{SL}(3, 3), \text{Spin}(5, 3) \).

**Conjecture.** As \( \mathcal{R} \)-modules, \( U_\lambda = R_\lambda \) if and only if \( \lambda \) is regular.

For the groups \( G_q \), one obtains (as in [5], [6]):

**Theorem B.** If \( \lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_k p^k \), define \( U_k = U_{\lambda_0} \otimes U_{\lambda_1} \otimes \cdots \otimes U_{\lambda_k} \) (as module for \( G \)). Then \( U_k \) is a projective \( \mathcal{R}_q \)-module (\( q = p_k + 1 \)), with \( R_\lambda \) as a direct summand.

**References**


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