WEAKLY CONTINUOUS ACCRETIVE OPERATORS

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We shall be concerned with the autonomous differential equation

\[ u'(t) + Au(t) = 0, \quad u(0) = x, \]

where \( A \) is a weakly continuous possibly nonlinear operator mapping a reflexive Banach space \( X \) to itself. Recently S. Chow and J. D. Schuur [2] have considered existence theory for ordinary differential equations involving weakly continuous operators on separable, reflexive Banach spaces.

We now make clear our notion of strong solutions to (1.1).

**Definition 1.2.** A function \( u : [0, T) \rightarrow X \) is said to be a strong solution to the Cauchy problem

\[ u'(t) + Au(t) = 0, \quad u(0) = x, \]

provided that \( u \) is Lipschitz continuous on each compact subset of \([0, T), u(0) = x, \ u \) is strongly differentiable almost everywhere and \( u'(t) + Au(t) = 0 \) for a.e. \( t \in [0, T). \)

By employing a variant of the Peano method we provide local solution to (1.1).

**Lemma 1.3.** Let \( X \) be a reflexive Banach space and suppose that \( A \) is a weakly continuous operator with \( D(A) = X \). Then there is a finite interval \([0, T)\) such that the Cauchy problem (1.1) has a strong solution on \([0, T). \)

**Definition 1.4.** An operator \( A \) is said to be accretive provided that

\[ \|x + \lambda Ax - (y + \lambda Ay)\| \geq \|x - y\| \]

for all \( \lambda \geq 0 \) and \( x, y \in D(A) \). T. Kato [5] has shown that this definition is equivalent to the statement that \( \text{Re}(Ax - Ay, f) \geq 0 \) for some \( f \in F(x - y) \) where \( F \) is the duality map from \( X \) to \( X^* \).

If we require that the operator \( A \) be accretive we are able to extend the local solution of Lemma 1.3 to a global solution.

**Theorem 1.5.** Let \( X \) be a reflexive Banach space and suppose that \( A \) is a weakly continuous accretive operator with \( D(A) = X \). Then the Cauchy problem (1.1) has a unique strong global solution on \([0, \infty). \)

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If we set \( u(t) = T(t)x \) we obtain a semigroup of nonlinear nonexpansive operators \( \{ T(t) : t \geq 0 \} \) which map \( X \) to \( X \). We can say that \( \{ T(t) : t \geq 0 \} \) is the semigroup associated with \( A \). The next theorem provides an exponential representation for \( \{ T(t) : t \geq 0 \} \).

**THEOREM 1.6.** Let \( A \) and \( X \) satisfy the conditions of Theorem 1.5. Then the operator \( A \) is \( m \)-accretive, i.e., \( R(I + \lambda A) = X \) for all \( \lambda \geq 0 \). If \( \{ T(t) : t \geq 0 \} \) is a semigroup associated with \( A \) then \( T(t) \) may be represented as the pointwise limit

\[
T(t)x = \lim_{n \to \infty} (I + t/nA)^n x.
\]

Moreover, for each fixed \( t_0 > 0 \), the operator \( T(t_0) \) is weakly continuous.

The \( m \)-accretiveness of \( A \) is obtained by considering the equation

\[
u'(t) + A'\nu(t) = 0
\]

where \( A' = A + I \). Once the \( m \)-accretiveness of \( A \) has been established the exponential representation of \( \{ T(t) : t \geq 0 \} \) follows immediately from a theorem of M. Crandall and T. Liggett [1]. The fact that \( T(t_0) \) is weakly continuous is obtained by showing that \( (I + \lambda A)^{-1} \) is weakly continuous for all \( \lambda \geq 0 \) and employing estimates of Crandall and Liggett. The foregoing results may be applied to the rest point theory developed by C. Yen [10].

**REFERENCES**