NONUNIFORMLY ELLIPTIC EQUATIONS: POSITIVITY OF WEAK SOLUTIONS

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1. This note is concerned with the weak boundary value problem

\[ \int_\Omega \left( \sum_{i,j=1}^N a_{ij}(x) u_{x_i} v_{x_i} + b(x) uv \right) dx = \int_\Omega c(x) f v dx, \quad \forall v \in C_0^\infty(\Omega), \]

and the weak eigenvalue problem

\[ \int_\Omega \left( \sum_{i,j=1}^N a_{ij}(x) u_{x_i} v_{x_i} + b(x) uv \right) dx = \lambda \int_\Omega c(x) uv dx, \quad \forall v \in C_0^\infty(\Omega), \]

where \( \Omega \) is a connected open set in \( \mathbb{R}^N \). Our hypothesis concerning the coefficient matrix \( (a_{ij}) \) in (1) and (2) is similar to but weaker than those imposed on the elliptic operators which are studied in [2], [3], [4]. Specifically, we assume that \( A = (a_{ij}) \) is a real matrix-valued function, symmetric and positive definite almost everywhere on \( \Omega \) with

\[ \|A\|, \|A^{-1}\| \in L^1_{\text{loc}}(\Omega). \]

Concerning the coefficients \( b, c \) our assumptions are the following: \( b \) and \( c \) are real valued,

\[ Mb \geq c > 0 \quad \text{a.e. on } \Omega \]

for some positive constant \( M \) and

\[ b, b^{-1}, c \in L^1_{\text{loc}}(\Omega). \]

Under these assumptions we prove: If \( f \in L^2(\Omega, c(x) dx), f(x) \geq 0 \) a.e. on \( \Omega \) and \( f \neq 0 \) then (1) has a solution positive almost everywhere on \( \Omega \), in particular a nonnegative eigenfunction of (2) is positive almost everywhere in \( \Omega \); if (2) has a nonnegative eigenfunction corresponding to an eigenvalue \( \lambda_1 > 0 \) then \( \lambda_1 \) is simple and the spectrum of (2) is contained in the interval \([\lambda_1, \infty)\].

This research was motivated by certain problems arising in connection with the study in [1] of nonlinear elliptic eigenvalue problems.

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2. We assume that \((a_{ij}), b\) and \(c\) are as above. Let \(X_1\) denote the set of functions \(u \in C^\infty(\Omega)\) for which

\[
\|u\|_{X_1}^2 = \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij} u_{x_i} u_{x_j} + bu^2 \right) \, dx < \infty,
\]

and let \(X_1\) be the Hilbert space obtained by completing \(X_1\) in the norm (6); \(X_0\) will denote the closure of \(C_0^\infty(\Omega)\) in \(X_1\).

**Lemma 1.** The space \(X_1\) is stronger than the space \(H^{1,1}_{\text{loc}}(\Omega)\), and the norm on \(X_1\) is given by (6). The spaces \(X_1\) and \(X_0\) are closed under the operation

\[
u \mapsto |\nu|;
\]

moreover, this operation is norm preserving in \(X_1\).

Here, as is standard, \(H^{1,1}_{\text{loc}}(\Omega)\) denotes the Fréchet space of locally integrable, locally strongly \(L^1\) differentiable functions on \(\Omega\).

Let \(X\) be any closed subspace of \(X_1\) with

\[
X_0 \subset X \subset X_1,
\]

and such that \(X\) is closed under the mapping (7), and let \(Y\) denote the Hilbert space consisting of measurable functions \(f\) for which

\[
\|f\|_Y^2 = \int_{\Omega} |f|^2 c(x) \, dx < \infty.
\]

From (4) and (6) it is clear that the functions in \(X\) are also in \(Y\). The inclusion mapping \(X \subset Y\) will be denoted by \(i\).

**Lemma 2.** The mapping \(i : X \to Y\) is bounded, injective, and has dense range. The mapping \(i^* : Y \to X\) (the Lax-Milgram operator) is also injective with dense range and preserves nonnegativity.

It is not difficult to see that when \(X = X_0\) then \(u = i^* f\), for \(f \in Y\), is the solution of (1).

We next consider the “Green’s operator” \(k = ii^*\) in \(Y\), and state the first of our two main results which refines the nonnegativity assertion of Lemma 2.

**Theorem 1.** The operator \(k\) is selfadjoint, positive definite, and bounded. If \(f\) is a nonzero element of \(Y\) and \(f(x) \geq 0\) a.e. on \(\Omega\) then \(h = kf\) satisfies

\[
h(x) > 0 \quad \text{a.e. on } \Omega.
\]

In particular if \(k\) has a nonnegative eigenfunction \(\varphi\), then

\[
\varphi(x) > 0 \quad \text{a.e. on } \Omega.
\]
REMARK. If $\Omega$ is bounded, $b = 0$, and the coefficients in (2) satisfy stronger regularity conditions then such a positivity result can be obtained from Lemma 2 and the Harnack inequality of Trudinger [4]; indeed in that case one can assert, instead of merely (9), that $h$ has a positive essential lower bound on each compact subset of $\Omega$. Our proof of Theorem 1 however makes use of global rather than local methods.

**Theorem 2.** Let $\varphi$ be a nonnegative eigenfunction of $k$, $\mu \varphi = k \varphi$, then $\|k\| = \mu$, and $\mu$ is a simple eigenvalue of $k$.

While Theorem 2 is very easily proved in the case where $k$ is compact, the general case is somewhat deeper and does not seem to be contained in the extensive literature on positive operators.

3. We now describe the sort of application of Theorems 1 and 2 which was wanted for [1]. We consider the problem (2) in $W^{1,p}_0(\Omega)$ for some $p$ with $2 \leq p \leq \infty$. With $p$ fixed we take

$$r = p/(p - 2)$$

and we take $s$ to be an element of the extended real number system with $p \leq s$ and $s \leq Np/(N - p)$, if $p \leq N$, finally we take

$$r_1 = s/(s - 2).$$

We assume that the matrix $A$ is as in §1 and in addition that

$$\|A\| \in L^r(\Omega).$$

Concerning $b$ and $c$ we assume only that

$$b, c \in L^s(\Omega).$$

**Theorem 3.** Let $u \in W^{1,p}_0(\Omega)$ be a nonnegative eigenfunction of the weak problem (2) corresponding to the eigenvalue $\lambda_1 > 0$. Then

$$u(x) > 0 \quad \text{a.e. in } \Omega,$$

and, for all $v \in W^{1,p}_0(\Omega)$,

$$\int_\Omega \left( \sum_{i,j=1}^N a_{ij} v_i v_j + bv^2 \right) dx \geq \lambda_1 \int_\Omega v^2 c(x) dx,$$

with equality only if $v$ is proportional to $u$.

**References**


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