RESEARCH ANNOUNCEMENTS

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INFINITE SUMS OF PSI FUNCTIONS

BY P. F. JORDAN
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A transformation. The reversible transformation, where \( \lambda = \lambda + \frac{1}{2} \),

\[
\pi C_{2\lambda+1} = 2 \sum_{\kappa=0}^{\infty} \frac{\lambda C_{2\kappa}}{\lambda^2 - \kappa^2} \quad \text{(all } \lambda),
\]

\[
\pi C_{2\kappa} = 2 \sum_{\lambda=0}^{\infty} \frac{\lambda C_{2\lambda+1}}{\lambda^2 - \kappa^2} \quad \text{(times } \frac{1}{2} \text{ if } \kappa = 0)
\]

has the properties[1]

\[
\sum_{\kappa=0}^{\infty} C_{2\kappa} = 0
\]

if the set \( C_{2\lambda+1} \) converges at least like \( \lambda^{-t}, t \geq 2 \), and

\[
S = \sum_{\lambda=0}^{\infty} (2\lambda + 1)C_{2\lambda+1} = 0
\]

if the set \( C_{2\kappa} \) converges at least like \( \kappa^{-r}, r > 2 \).

Consider in particular the elementary sets

\[
C_0 = \zeta(r), \quad C_{2\kappa+1} = -\kappa^{-r} \quad (r = 2, 3, 4, \ldots)
\]

which obey (2), and

\[
C_{2\lambda+1} = \lambda^{-t} \quad (t = 2, 3, 4, \ldots).
\]

For \( r = 2 \), the sum \( S \) is \( S_2 = \pi \).

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Applying the transformations (1a, b) to the elementary sets (4a, b), one has
\[ \pi C_{2\lambda+1} = + \sum_{t=3,5,7,...} (A_t'/\lambda') \]
with
\[
A_t' = \begin{cases} 
-2\zeta(r + 1 - t) & \text{if } t < r \\
-4L_{\lambda}^* & \text{if } t = r \quad (r \text{ odd only}) \\
+1 & \text{if } t = r + 1 \quad (r \text{ even only}) \\
0 & \text{if } t > r + 1 
\end{cases}
\]
and, using the abbreviation \( \zeta(n) = \sum_{\lambda=0}^{\infty} (1/\lambda^n) = (2^n - 1) \zeta(n) \), one has
\[ \pi C_{1} = \zeta(t + 1), \quad \pi C_{2\lambda+1} = - \sum_{r=2,4,6,...} (B_r'/\kappa') \]
with
\[
B_t' = \begin{cases} 
2\zeta(t + 1 - r) & \text{if } r < t \\
4L_{\lambda} & \text{if } r = t \quad (t \text{ even only}) \\
0 & \text{if } r > t.
\end{cases}
\]

Here \( L \) and \( L^* \) represent the \( \psi \)-function and may hence be denoted as logarithmic sets:
\[ L_m = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m-1} = \frac{1}{2} [\psi(m + \frac{1}{2}) - \psi(\frac{1}{2})] \]
\[ \sim \frac{1}{2} (\log m + \gamma) + \log 2 + 1/48m^2 + \cdots \]
\[ L_n^* = L_m + 1/2(2m + 1) - \log 2. \]

\textbf{Infinite sums.} As one applies the summation (2) to the sets \( C_{2\lambda} \), one regains the known values \( \zeta(2n) \) (see [2, 23.2.16]), if \( t \) is odd, that is, when \( C_{2\lambda} \) does not contain a logarithmic set. On the other hand, an intriguing sequence of new formulas arises when \( t \) is even. For \( t = 2 \),
\[ 4 \sum_{k=1}^{\infty} L_k/k^2 = 7\zeta(3). \]
The general formula is
\[ 4 \sum_{k=1}^{\infty} L_k/k^{2n} = \zeta(2n + 1) - 2 \sum_{v=1}^{n-1} \zeta(2v)\zeta(2n + 1 - 2v). \]
In a sense, this formula can be considered a formula for \( \zeta(2n + 1) \) which
corresponds to the known formula [2, 23.2.16] for $\zeta(2n)$.

A companion sequence of formulas arises from forming the sum $S$ for the sets $C_{2\lambda+1}^+$. In this case one recovers the known values $\zeta(2n)$ if $r$ is even. For $r = 3$,

\begin{equation}
16 \sum_{k=1}^{\infty} \frac{L_k}{(2k - 1)^2} = 7\zeta(3) + 12\zeta(2) \log 2
\end{equation}

and generally

\begin{equation}
4^{n+1} \sum_{k=1}^{\infty} \frac{L_k}{(2k - 1)^{2n}} = \bar{\zeta}(2n + 1) + 4\zeta(2n) \log 2 - 2 \sum_{v=1}^{n-1} \zeta(2v)\zeta(2n + 1 - 2v).
\end{equation}

The two sequences of formulas have considerable similarity. Both are homogeneous in the sum of the arguments in each term if $\log 2$ is written as $\eta(1)$ (see [2, 23.2.19]).

NOTE. The subject reversible transformation arises in the linear theory of a parabolic wing tip in lifting subsonic flow. The fact that it may produce logarithmic sets can be generalized, as follows: If the originating set contains $\log^n$, the logarithmic set in the transformed set is $\log^{n+1}$ if $r$ is odd and if $t$ is even, and is $\log^{n-1}$ if $r$ is even and if $t$ is odd. Detailed derivations are given in [1].

REFERENCES


MARTIN MARIETTA LABORATORIES, BALTIMORE, MARYLAND 21227