SELFADJOINT SUBSPACE EXTENSIONS OF NONDENSELY DEFINED SYMMETRIC OPERATORS

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1. Subspaces in \( \mathcal{H}^2 \). Let \( \mathcal{H} \) be a Hilbert space over the complex field \( \mathbb{C} \), and let \( \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H} \) be the Hilbert space of all pairs \( \{f, g\} \), where \( f, g \in \mathcal{H} \), with the inner product \( \langle \{f, g\}, \{h, k\} \rangle = (f, h) + (g, k) \). A subspace \( T \) in \( \mathcal{H}^2 \) is a closed linear manifold in \( \mathcal{H} \); its domain \( \mathcal{D}(T) \) is the set of all \( f \in \mathcal{H} \) such that \( \{f, g\} \in T \) for some \( g \in \mathcal{H} \), and its range \( \mathcal{R}(T) \) is the set of all \( g \in \mathcal{H} \) such that \( \{f, g\} \in T \) for some \( f \in \mathcal{H} \). For \( f \in \mathcal{D}(T) \) we put \( T(f) = \{g \in \mathcal{H} | \{f, g\} \in T\} \). A subspace \( T \) in \( \mathcal{H}^2 \) is the graph of a linear function if \( T(0) = \{0\} \); in this case we say \( T \) is an operator in \( \mathcal{H} \), and then we denote \( T(f) \) by \( Tf \).

The adjoint \( T^* \) of a subspace \( T \) in \( \mathcal{H}^2 \) is defined by

\[ T^* = \{\{h, k\} \in \mathcal{H}^2 | (g, h) = (f, k) \text{ for all } \{f, g\} \in T\}. \]

If \( J \) is the unitary operator in \( \mathcal{H}^2 \) given by \( J\{f, g\} = \{g, -f\} \), then \( T^* = \mathcal{H}^2 \ominus JT \), the orthogonal complement of \( JT \) in \( \mathcal{H}^2 \). This shows that \( T^* \) is also a subspace in \( \mathcal{H}^2 \).

If \( T \) is a subspace in \( \mathcal{H}^2 \), let \( T_\alpha = \{\{f, g\} \in T | f = 0\} \). Then \( T_\alpha = (T \ominus T_\alpha) \), is a closed operator in \( \mathcal{H} \), and we have the orthogonal decomposition \( T = T_s \oplus T_\alpha \), with \( \mathcal{D}(T_s) \) dense in \( \mathcal{H} \ominus T^*(0) \), \( \mathcal{R}(T_s) \subset \mathcal{H} \ominus T(0) \).

A symmetric subspace \( S \) in \( \mathcal{H}^2 \) is one satisfying \( S \subseteq S^* \), and a selfadjoint subspace \( H \) is a symmetric one such that \( H = H^* \). If \( H = H_s \oplus H_\alpha \) is a selfadjoint subspace in \( \mathcal{H}^2 \) we have the result (due to Arens, [1, Theorem 5.3]) that \( H_s \), considered as an operator in \( \mathcal{H} \ominus H(0) \), is a densely defined selfadjoint operator there. This permits a spectral analysis of a selfadjoint subspace \( H \), once its operator part \( H_s \) and its purely multi-valued part \( H_\alpha \) have been identified.

If \( S, S_1 \) are symmetric subspaces in \( \mathcal{H}^2 \) such that \( S \subseteq S_1 \), then \( S_1 \) is said to be a symmetric extension of \( S \). In [3] (see also [2]) we described all symmetric and selfadjoint extensions of a symmetric subspace \( S \) in \( \mathcal{H}^2 \). In this note we characterize precisely, in terms of “generalized boundary conditions”, those selfadjoint subspace extensions of a non-densely defined symmetric operator \( S \) in \( \mathcal{H} \). Applications to ordinary differential operators will be indicated in a subsequent note. Detailed

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propositions will appear elsewhere.

We require from [3, Theorems 12 and 15] two characterizations of the selfadjoint extensions $H$ of a symmetric subspace $S$ in $\mathcal{H}^2$. All such satisfy $S \subset H \subset S^*$; let $M = S^* \ominus S$.

**Theorem A.** A subspace $H$ is a selfadjoint extension of $S$ in $\mathcal{H}^2$ if and only if $H = S \oplus M_1$, where $M_1$ is a subspace of $M$ satisfying $J M_1 = M \ominus M_1$.

Alternatively, such $H$ may be described in terms of the subspaces $M = S^* \ominus S$. We have $M = M^+ \oplus M^-$ and the following result.

**Theorem B.** A subspace $H$ is a selfadjoint extension of $S$ in $\mathcal{H}^2$ if and only if there exists an isometry $V$ of $M^+ \rightarrow M^-$ such that $H = S^* \ominus (I - V) M^+$, where $I$ is the identity operator. Thus $S$ has a selfadjoint extension in $\mathcal{H}^2$ if and only if $\dim M^+ = \dim M^-$.

2. **Selfadjoint extensions of nondensely defined symmetric operators.**

Let $S_0$ be a symmetric densely defined operator in $\mathcal{H}$, and let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$. Throughout this section we assume that

\begin{equation}
\dim \mathcal{H}_0 = p < \infty, \quad \dim M_0 < \infty, \quad M_0 = S_0^* \ominus S_0.
\end{equation}

We define $S$ to be the operator in $\mathcal{H}$ given by

\begin{equation}
\mathcal{D}(S) = \mathcal{D}(S_0) \cap (\mathcal{H} \ominus \mathcal{H}_0), \quad S \subset \mathcal{H}_0.
\end{equation}

This operator is not densely defined, and so its adjoint will be a subspace which is not an operator.

**Theorem 1.** Let $S$ be defined by (2.2), where (2.1) is assumed. Then $S$ is a symmetric operator with $\mathcal{D}(S)$ dense in $\mathcal{H} \ominus \mathcal{H}_0$, and

\begin{equation}
S^* = \{\{h, S_0^* h + \phi\}| h \in \mathcal{D}(S_0^*), \phi \in \mathcal{H}_0\},
\end{equation}

\begin{equation}
\dim M^\pm = \dim (M_0)^\pm + \dim \mathcal{H}_0.
\end{equation}

Thus $S^*(0) = \mathcal{H}_0$ and $S^*$ is the algebraic sum of $S_0^*$ and $(S^*)_\infty$. From (2.4) and Theorem B it follows that $S$ has selfadjoint extensions in $\mathcal{H}^2$ if and only if $\dim (M_0)^+ = \dim (M_0)^-$, that is, if and only if $S_0$ has selfadjoint extensions in $\mathcal{H}$. We now assume $\dim (M_0)^+ = \dim (M_0)^- = \omega$, and indicate how one can characterize any selfadjoint extension $H$ of $S$ in $\mathcal{H}^2$ by means of "generalized boundary conditions". Theorem A implies that any such $H = S \ominus M_1$ can be thought of as $H = S^* \ominus J M_1$, where $\dim M_1 = p + \omega$. Thus

\begin{equation}
H = \{\{h, k\} \in S^*| (k, \alpha) - (h, \beta) = 0 \text{ for all } \{\alpha, \beta\} \in M_1\},
\end{equation}

and (2.3) implies that $H$ is the set of all $\{h, S_0^* h + \phi\} \in S^*$ satisfying
(2.5) \[ \langle h\alpha \rangle - (h, \varphi') + (\varphi, \alpha) = 0 \]

for all \( \{\alpha, S_0^* \alpha + \varphi'\} \in M_1 \). Here we have introduced the abbreviation
\( \langle h\alpha \rangle = \langle S_0^* h, \alpha \rangle - (h, S_0^* \alpha), h, \alpha \in \mathcal{D}(S_0^*) \). By thinking of \( H \) as in (2.5) we obtain the following precise characterization.

**Theorem 2.** Let \( H \) be a selfadjoint subspace extension of \( S \) in \( S^2 \), with \( \dim H(0) = s \). Let an orthonormal basis for \( H(0) \) be \( \varphi_1, \ldots, \varphi_s \), and suppose \( \varphi_1, \ldots, \varphi_s, \varphi_{s+1}, \ldots, \varphi_p \) is an orthonormal basis for \( S^*(0) = S_0 \). Then \( H \) is the set of all \( \{h, S_0^* h + \varphi \} \in S^* \) such that

(i) \( (h, \varphi_j) = 0, j = 1, \ldots, s \),
(ii) \( \langle h\delta_{jk} \rangle - (h, \zeta_j) = 0, j = p + 1, \ldots, p + \omega \),
(iii) \( \varphi = c_1 \varphi_1 + \cdots + c_s \varphi_s + \sum_{k=s+1}^p [\langle h, \psi_k \rangle - \langle h\gamma_k \rangle] \varphi_k, c_j \in \mathbb{C} \) arbitrary,

where

(a) \( \gamma_{s+1}, \ldots, \gamma_p \in \mathcal{D}(S_0^*) \),
(b) \( \delta_{p+1}, \ldots, \delta_{p+\omega} \in \mathcal{D}(S_0^*) \) are linearly independent mod \( \mathcal{D}(S_0) \), and
(c) \( \zeta_j = -\sum_{k=s+1}^p \langle \delta j^k \rangle \varphi_k, j = p + 1, \ldots, p + \omega \),
(d) \( \psi_j = \sum_{k=s+1}^p [E_{kj} - \frac{1}{2} \langle \gamma j^k \rangle] \varphi_k, j = s + 1, \ldots, p \), \( E_{kj} \in \mathbb{C}, E = (E_{jk}) \) = \( E^* \).

Conversely, let \( \varphi_1, \ldots, \varphi_p \) be an orthonormal basis for \( S_0 \), suppose \( \gamma_j, \delta_j \) exist satisfying (a), (b), and \( \zeta_j, \psi_j \) are defined by (c), (d). Then \( H \) defined via (i)–(iii) is a selfadjoint extension of \( S \) with \( \dim H(0) = s \).

The operator part \( H_s \) of \( H \) is given by

\[ H_s h = Q_0 S_0^* h + \sum_{k=s+1}^p [\langle h, \psi_k \rangle - \langle h\gamma_k \rangle] \varphi_k, \]

where \( Q_0 \) is the orthogonal projection of \( S \) onto \( \mathcal{D}(S_0^*) \).

With appropriate interpretations, Theorem 2 remains valid in the three cases: \( s = 0, s = p, \) and \( \omega = 0 \). If \( s = 0 \) then \( H \) is an operator extension of \( S \), and those operator extensions \( H \) satisfying \( S_0 \subset H \subset S_0^* \) are obtained by taking \( \gamma_j = 0, E_{kj} = 0 \), which results in \( \zeta_j = 0, \psi_j = 0 \). Then

\[ \mathcal{D}(H) = \{h \in \mathcal{D}(S_0^*), \langle h\delta_j \rangle = 0, j = p + 1, \ldots, p + \omega \}, \]

which is the known characterization of such \( H \). If \( \omega = 0 \) and \( s = p \), \( H(0) = S_0 \) and \( H_s h = Q_0 S_0 h \). Thus, given any selfadjoint operator \( S_0 \) in \( S_0 \), with \( \mathcal{D}(S_0) \) dense in \( S_0 \), and subspace \( S_0 \subset S_0 \), dim \( S_0 < \infty \), the operator \( H_s \) on \( S_0 \oplus S_0 \) defined by \( H_s h = Q_0 S_0 h \) is a densely defined selfadjoint operator. This is a result due to W. Stenger [4, Lemma 1].

**References**


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