A COMPLETE BOOLEAN ALGEBRA OF SUBSPACES WHICH IS NOT REFLEXIVE

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This note provides a negative answer to a question raised by P. R. Halmos [2, Problem 9]. For the convenience of the reader, the terminology necessary to understand the question is presented here. Let $\mathcal{L}$ be a lattice of subspaces of a Hilbert space $\mathcal{H}$ and let $\text{Alg}\, \mathcal{L}$ be the algebra of all bounded operators in $\mathcal{B}(\mathcal{H})$ that leave each subspace in $\mathcal{L}$ invariant. If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, let $\text{Lat}\, \mathcal{A}$ be the lattice of all subspaces of $\mathcal{H}$ that are left invariant by each operator in $\mathcal{A}$. A lattice $\mathcal{L}$ is reflexive if $\text{Lat}\, \text{Alg}\, \mathcal{L} = \mathcal{L}$. If $\mathcal{L}$ is a reflexive lattice and $\{P_i\}$ is a net of orthogonal projections such that $P_i(\mathcal{H}) \in \mathcal{L}$ for each $i$ and $P_i \to P$ in the strong operator topology then $P(\mathcal{H}) \in \mathcal{L}$; in other words, $\mathcal{L}$ is strongly closed. It is true that a strongly closed lattice of subspaces is a complete lattice, but the converse is false.

A Boolean algebra of subspaces is a distributive lattice $\mathcal{L}$ such that for each $M$ in $\mathcal{L}$ there is a unique $M'$ in $\mathcal{L}$ such that $M \cap M' = (0)$ and $M \vee M' \equiv (M + M')^\perp = \mathcal{H}$. (Note that it is only required that $\mathcal{H}$ be the closure of $M + M'$.) Problem 9 of [2] asks: Is every complete Boolean algebra of subspaces reflexive? The answer is no, and this is shown in this paper by giving a complete Boolean algebra of subspaces which is not strongly closed. In one sense this answer seems unsatisfactory because a new question arises: Is every strongly closed Boolean algebra of subspaces reflexive? In another sense the answer is satisfying because the original question was the proper one to ask. The property of completeness is a lattice theoretic one, while the property of being strongly closed is not.

For the remaining terminology the reader is referred to [4] and other standard references. If $X = [0, 2\pi]$, let $\mu$ be a positive singular measure on the collection $\mathcal{A}$ of Borel subsets of $X$. For $A$ in $\mathcal{A}$ define

$$\varphi_A(z) = \exp \left( -\int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right), \quad |z| < 1,$$

and put $\varphi = \varphi_X$. Each $\varphi_A$ is an inner function, and $\varphi_A$ is a divisor of $\varphi_B$ if and only if $A \subset B$. $\mathcal{H} = H^2 \ominus \varphi H^2$ and, for each $A$ in $\mathcal{A}$, $M_A = \varphi_A H^2 \ominus \varphi H^2$. 

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(1) \[ M_A \cap M_B = M_{A \cup B}. \]

In fact, \( \varphi_A H^2 \cap \varphi_B H^2 = \psi H^2 \) where \( \psi \) is the least common multiple of \( \varphi_A \) and \( \varphi_B \). It is easy to check that \( \psi = \varphi_{A \cup B} \) and from this it follows that (1) holds. Similarly

(2) \[ M_A \vee M_B = M_{A \cap B}. \]

It follows from (1) and (2) that \( \mathcal{L} = \{M_A : A \in \mathcal{A}\} \) is a distributive lattice; and, with \( M'_A = M_{X - A} \), \( \mathcal{L} \) is a Boolean algebra of subspaces of \( \mathcal{H} \).

**Lemma.** \( \mathcal{L} = \{M_A : A \in \mathcal{A}\} \) is a complete Boolean algebra.

**Proof.** It suffices to show that if \( \mathcal{B} \subset \mathcal{A} \) then there is an \( A \) in \( \mathcal{A} \) with \( M_A = \bigcap \{M_B : B \in \mathcal{B}\} \). Because of (1), \( \mathcal{B} \) may be assumed to be closed under finite unions. If \( \beta = \sup \{\mu(B) : B \in \mathcal{B}\} \) then there is an increasing sequence \( \{B_n\} \) in \( \mathcal{B} \) such that \( \beta = \lim_n \mu(B_n) \). If \( A = \bigcup \{B_n : n \geq 1\} \) then \( \mu(A) = \beta \) and \( \mu(B - A) = 0 \) for every \( B \) in \( \mathcal{B} \). It is claimed that \( \varphi_A = \text{l.c.m.} \{\varphi_B : B \in \mathcal{B}\} \). In fact, if \( B \in \mathcal{B} \) then \( \varphi_A = \varphi_{A - B} \varphi_B \) since \( \mu(B - A) = 0 \). Also, if \( \psi \) is an inner function that is a multiple of \( \varphi_B \) for each \( B \) in \( \mathcal{B} \) then, for every integer \( n \), \( \psi = \varphi_{B_n} \psi_n \) for some inner function \( \psi_n \). But \( \varphi_{B_n}(z) \to \varphi_A(z) \) for every \( z \) so it follows that \( \psi_n(z) \to \psi(z) \) for some inner function \( \psi \). Hence \( \psi = \varphi_A \psi \) and \( \varphi_A = \text{l.c.m.} \{\varphi_B : B \in \mathcal{B}\} \). Consequently,

\[ M_A = \bigcap \{M_B : B \in \mathcal{B}\}. \]

**Theorem.** \( \mathcal{L} = \{M_A : A \in \mathcal{A}\} \) is reflexive if and only if \( \mu \) is a purely atomic measure.

**Proof.** If \( \mu \) is purely atomic then \( \mathcal{L} \) is an atomic Boolean algebra and hence is reflexive [3]. To prove the converse, two additional results are needed. The first of these can be found in [5] although the proof contains an error (which can be rectified). However, in the case under consideration (where \( L^1(\mu) \) is separable) the proof is valid. (Also see [1].)

**Theorem A.** Let \( (X, \mathcal{A}, \mu) \) be a decomposable nonatomic measure space and let \( f \in L^\infty(X, \mathcal{A}, \mu) \) such that \( 0 \leq f \leq 1 \). Then there is a sequence \( \{A_n\} \) in \( \mathcal{A} \) such that \( \chi_{A_n} \to f \) in the weak-star topology of \( L^\infty \).

**Theorem B.** For each inner function \( q \) let \( E_q \) be the orthogonal projection of \( H^2 \) onto \( qH^2 \). If \( q, q_1, q_2, \ldots \) are inner functions such that \( q(z) = \lim_n q_n(z) \) for \( |z| < 1 \) then \( E_{q_n} \to E_q \) strongly.

**Proof.** If \( z^n \) is the function that assumes the value \( a_m \) at \( a \) then it is easily verified that
\[ E_q(z^m) = q \sum_{k=0}^{m} \frac{1}{k!} q^{(k)}(0) z^{m-k}. \]

It follows that \( E_q(z^m)(a) = \lim_n E_{q_n}(z^m)(a) \) for \(|a| < 1\). This gives that \( E_{q_n}(z^m) \to E_q(z^m) \) weakly in \( H^2 \). Since polynomials are dense in \( H^2 \), \( E_{q_n} \to E_q \) in the weak operator topology. But for projections weak convergence is equivalent to strong convergence, and the proof is complete.

Suppose \( \mu \) is not purely atomic; the proof of the main theorem will be completed by showing that \( L \) is not strongly closed. There is a set \( A \) in \( \mathcal{A} \) that contains no atoms for \( \mu \) and with \( \mu(A) > 0 \). Let \( f \) be any Borel function such that \( 0 \leq f \leq 1 \), \( f(x) = 0 \) for \( x \) in \( X - A \), and \( 0 < f(x) < 1 \) on a set of positive measure. According to Theorem A there is a sequence \( \{A_n\} \) in \( \mathcal{A} \) such that \( A_n \subset A \) and \( \chi_{A_n} \to f \) in the weak-star topology of \( L^\infty(\mu) \). For each \( z, |z| < 1 \),

\[ \varphi_{A_n}(z) \to \psi(z) = \exp \left( - \int \frac{e^{i\theta} + z}{e^{i\theta} - z} f(\theta) \, d\mu(\theta) \right). \]

Theorem B implies that \( E_{\varphi_{A_n}} \to E_\psi \) strongly; so \( E_{\varphi_{A_n}} - E_\varphi \to E_\psi - E_\varphi \) strongly. It is straightforward to show that if \( P_A \) is the projection of \( \mathcal{H} \) onto \( M_A \), then \( P_{A_n} \to P_\psi \), where \( P_\psi \) is the projection of \( \mathcal{H} \) onto \( \psi H^2 \ominus \varphi H^2 \).

Since \( \psi H^2 \ominus \varphi H^2 \neq M_A \) for any \( A \), the proof is complete.

Finally, it should be pointed out that \( L \) is isomorphic to the reflexive Boolean algebra \( \text{Lat} T \), where \( T \) is multiplication by the independent variable on \( L^2(X, \mu) \).

REFERENCES


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