A SMASH PRODUCT FOR SPECTRA

BY HAROLD M. HASTINGS

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ABSTRACT. We shall show that the smash product for pointed CW complexes induces a smash product \( \wedge \) on the homotopy category of Adams's stable category with the following properties: \( \wedge \) is coherently homotopy unitary (\( S^0 \)), associative, and commutative, \( \wedge \) commutes with suspension up to homotopy, and \( \wedge \) satisfies a Kunneth formula.

Introduction. Precisely, we shall show that the homotopy category of a technical variant of Adams's stable category [1], a fraction category of CW prespectra equivalent to that of Boardman [3], [8], admits a symmetric monoidal structure in the sense of [4].

Whitehead's pairings of prespectra [9] and Kan and Whitehead's nonassociative smash product for simplicial spectra [6] were the first attempts at a smash product. Boardman gave the first homotopy associative, commutative, and unitary smash product in his stable category [3]. Adams has recently obtained a similar construction [2].

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1. The interchange problem. A \( S^k \)-prespectrum \( X \) consists of a sequence of pointed spaces \( \{X_n \mid n \geq 0\} \), together with inclusions \( X_n \wedge S^k \to X_{n+1} \). Consider \( S = \{S_n = S^{nk}\} \) as a ring with respect to the smash product of spaces. Then \( X \) is a right \( S \)-module.

Construction of a homotopy associative smash product for \( S^k \)-prespectra requires permutations \( \pi \) of \( S^k \wedge \cdots \wedge S^k \). Since \( S \) is not strictly commutative, but only graded homotopy commutative, this requires defining suitably canonical maps of degree \(-1\) (for \( k \) odd) and homotopies \( \pi \simeq \pm \text{id} \).

We avoid sign problems by using \( S^4 \)-prespectra and define canonical homotopies \( H_\pi \) as follows. Make the standard identifications

\[ S^{4k} \cong S^4 \wedge \cdots \wedge S^4 \cong I^4/\partial I^4 \wedge \cdots \wedge I^4/\partial I^4 \cong I^{4k}/\partial I^{4k} \cong D^{4k}/S^{4k-1} \, . \]

Then \( \pi \) simply permutes factors of \( C^2 \times \cdots \times C^2 \). Hence \( \pi \in SU(2k) \).


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Since $SU(2k)$ is path connected and simply connected, there is a unique homotopy class of paths $[\Gamma_n]$ in $SU(2k)$ with $\Gamma_n(0) = \pi$, $\Gamma_n(1) = e$. Define homotopies $H_\pi : S^{4k} \wedge I^* \to S^{4k}$ by $H_\pi(s, t) = \Gamma_n(t)(s)$. Then $[H_\pi]$ is the required homotopy class (rel the endpoints) of canonical homotopies.

**Theorem 1.** Let $\pi$ and $\pi'$ be permutations of $S^{4k}$. Let $H_\pi$ and $H_{\pi'}$ be canonical homotopies. Then $H_\pi(H_{\pi'}(s, t), t) : S^{4k} \wedge I^* \to S^{4k}$ is a canonical homotopy for $\pi'\pi$.

Adams solves the interchange problem directly for $S^1$-prespectra by an argument involving special classes of paths in $SO$.

2. The Adams completion. Give $S^4$ the CW structure with one 4-cell and one 0-cell. Give $[0, n]$ the CW structure with 0-cells $0, 1, \ldots, n$.

**Definition 2.** A prespectrum consists of a sequence of pointed CW complexes $\{X_n | n \geq 0\}$, together with inclusions as subcomplexes $X_n \wedge S^4 \to X_{n+1}$. A strict (weak) map of prespectra $f : X \to Y$ consists of a sequence of continuous maps $\{f_n : X_n \to Y_n\}$ such that $f_{n+1} = f_n \wedge S^4$ (resp. $f_{n+1} \simeq f_n \wedge S^4$) on $X_n \wedge S^4$. $Ps$ is the category of prespectra and strict maps.

**Definition 3 (Adams).** A subspectrum $X' \subset X$ is cofinal if for each cell $\sigma$ of $X$ a sufficiently high suspension of $\sigma$ is in $X'$.

**Definition 4 (Adams).** $Ad$ is the category of prespectra in which maps from $X$ to $Y$ are diagrams $X \to X' \to Y$ in $Ps$ with $X'$ cofinal in $X$.

Formally, $Ad$ is the right fraction category $[5]$ of $Ps$ in which cofinal inclusions are invertible. Morphisms $X \to X' \xrightarrow{f} Y$ and $X \to X'' \xrightarrow{g} Y$ are equal if $f = g$ on $X' \cap X''$.

**Definition 5.** Let $X$ be a prespectrum. Given a monotone unbounded sequence of nonnegative integers $\{j_n | n \geq 0, j_n \leq n\}$, define a prespectrum $DX$ by $(DX)_n = X_{j_n} \wedge S^{4(n-j_n)}$, with inclusions $(DX)_n \wedge S^4 \to (DX)_{n+1}$ defined so that $DX \subset X$.

Then $D$ extends to a functor (destabilization) on $Ps$, and there are natural cofinal inclusions $DX \subset X$.

There are smash products $\wedge : Ps, CW \to Ps$, and $\wedge : Ad, CW \to Ad$; these are defined degreewise.

**Definition 6.** Maps $f, g : X \to Y$ in $Ps (Ad)$ are homotopic if there is a map $H : X \wedge I^* \to Y$ in $Ps (resp. Ad)$ with $H|X \wedge 0^* = f, H|X \wedge 1^* = g$.

Homotopy has the usual properties. Denote the resulting homotopy categories $Ht(PS), Ht(Ad)$.

3. Smash products. We shall define a family of smash products $\wedge$ on $Ht(Ad)$. 
DEFINITION 7. Let $X$ and $Y$ be prespectra. Given a sequence of pairs of nonnegative integers $\{(i_n, j_n) \mid n \geq 0, i_n + j_n = n, \}$ and $\{i_n\}$ and $\{j_n\}$ are monotone unbounded sequences}, let $X \wedge Y$ be the prespectrum with $(X \wedge Y)_n = X_{i_n} \wedge Y_{j_n}$; the required inclusions are induced from $X$ and $Y$.

Then $\wedge$ extends successively to bifunctors on $P_S$, $Ad$ (since the smash product of cofinal inclusions is cofinal) and $Ht(Ad)$.

4. Uniqueness and the symmetric monoidal structure.

DEFINITION 8. Let $X$ be a prespectrum. A permutation $\Pi$ of $DX$ consists of a sequence of maps $\Pi_n = X_{j_n} \wedge \pi_n : (DX)_n = X_{j_n} \wedge S^4 \wedge \cdots \wedge S^4 \to DX_n$, where each $\pi_n$ is a permutation of $S^4 \wedge \cdots \wedge S^4$. If $g = \{g_n : (DX)_n \to Y_n \mid g_{n+1}$ extends $g_n \wedge S^4$ up to permutation}, call $g$ a permutation map.

PROPOSITION 9. Permutation maps are weak maps, where the required homotopies $H_n : (DX)_n \wedge S^4 \wedge I^* \to Y_{n+1}$ are induced by canonical homotopies ($\S 1$).

Also, permutation maps may be destabilized and are closed under the following composition: $f : DX \to Y$ and $g : D'Y \to Z$ yield $gD(f) : D'DX \to D'Y \to Z$. Two permutation maps $f, g : DX \to Z$ differ by a permutation if for some permutation $\Pi$ of $DX$, $g = f\Pi$.

THEOREM 10. Any two smash products $\wedge$ and $\wedge'$ on $Ht(Ad)$ are naturally isomorphic.

PROOF. There are natural destabilizations and permutation classes of permutation maps $D(X \wedge Y) \to X \wedge' Y, D'(X \wedge' Y) \to X \wedge Y$. The composites $D'D(X \wedge Y) \to X \wedge Y$ and $DD'(X \wedge' Y) \to X \wedge' Y$ differ from the respective (cofinal) inclusions by permutations. Thus it suffices to prove the following.

LEMMA 11. A permutation commutative diagram of permutation maps $DX \to Y$ induces a commutative diagram in $Ht(Ad)$.

We shall sketch a proof in $\S 6$.

THEOREM 12. There are natural maps in $Ht(Ad)$,

\[
\begin{align*}
X & \to X \wedge S^0 \to X \\
\alpha : (X \wedge Y) \wedge Z & \to X \wedge (Y \wedge Z) \\
\beta : X \wedge Y & \to Y \wedge X
\end{align*}
\]

which form a symmetric monoidal category.

PROOF. There are natural destabilizations and permutation classes of permutation maps $DX \to X \wedge S^0 \to X$,

\[
\alpha' : D(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z),
\]
and

\[ c': D(X \wedge Y) \to Y \wedge X. \]

By Lemma 11, it suffices to obtain the coherency diagrams \([4]\) as permutation-commutative diagrams of permutation maps involving destabilizations. For example, the diagram for coherency of associativity is

\[
\begin{array}{c}
D^2((W \wedge X) \wedge Y) \wedge Z) \\
\xrightarrow{D^2(a')} \\
D^2(W \wedge (X \wedge (Y \wedge Z))) \\
\xrightarrow{D^2(a)} \\
D^2((W \wedge (X \wedge Y)) \wedge Z) \\
\xrightarrow{D^2(b)} \\
D(W \wedge (X \wedge (Y \wedge Z))) \\
\end{array}
\]

where \( D \) is used generically, and \((a' \wedge Z)_\ast\) and \((W \wedge a')_\ast\) are representatives of permutation classes of permutation maps induced by \( a' \).

These diagrams are readily obtained.

5. Telescopes. We sketch the main properties of the following telescope construction.

**Definition 13.** For a prespectrum \( X \), let \( \text{Tel} \ X \) be the prespectrum with \((\text{Tel} \ X)_n = \bigcup_{j=0}^{n} (X_j \wedge S^{4(n-j)} \wedge [j, n]^\ast)\), the iterated mapping cylinder of \( X_0 \wedge S^{4n} \to \cdots \to X_{n-1} \wedge S^4 \to X_n \), together with the induced inclusions.

Then \( \text{Tel} \) may be extended to a functor, and there are natural projections \( p_X: \text{Tel} \ X \to X \).

**Proposition 14.** \( p_X \) admits a homotopy inverse \( s_X \).

**Proof.** To define \( s_X \), show that \( \text{Tel} \ X \) and \( X \) are strong deformation retracts of the prespectrum \( Y \) with \( Y_n = X_n \wedge [0, n]^\ast \) such that \( p_X \) is the composite \( \text{Tel} \ X \to Y \to X \).

**Proposition 15.** A weak map \( f: X \to Y \), together with a family of homotopies for \( f \), \( \{H_n: X_n \wedge S^4 \to Y_{n+1}\} \), from \( f_n \wedge S^4 \to f_{n+1} | X_n \wedge S^4 \), induces strict maps \( \phi: \text{Tel} \ X \to \text{Tel} \ Y \) and \( F: X \to Y \).

**Proof.** Let \( \phi_0 = f_0 \), and for \( n \geq 0 \), define \( \phi_{n+1} \) by

\[
\phi_{n+1}(x, t) = (\phi_n \wedge S^4)(x, t) \quad \text{for } t \leq n,
\]

\[
= ((f_n \wedge S^4)(x), 2t - n) \quad \text{for } n \leq t \leq n + \frac{1}{2},
\]

\[
= (H_n(x, 2t - 2n - 1), n + 1) \quad \text{for } n + \frac{1}{2} \leq t \leq n + 1.
\]

Also, let \( F = p_Y \phi s_X \).

**Proposition 16.** Let \( \{H_n\} \) and \( \{H'_n\} \) be homotopies for a weak map \( f: X \to Y \), and assume that \( H_n \simeq H'_n \) rel the endpoints for all \( n \). Then
(f, \{H_n\}) and (f, \{H'_n\}) induce homotopic strict maps \text{Tel} X \to \text{Tel} Y and \(X \to Y\).

**Proposition 17.** Let \(f': X \to Y\) and \(f'': Y \to Z\) be weak maps with homotopies \(\{H_n\}\) and \(\{H'_n\}\), respectively. Then there are composed homotopies \(\{H_n\}\) for \(f''f'\) such that for the respective induced strict maps \(F': X \to Y, F'': Y \to Z,\) and \(F: X \to Z, F \simeq F''F'\).

6. **Proof of Lemma 11.** Given a permutation-commutative square in which each map and composite (see §4) is a permutation map,

\[
\begin{array}{ccc}
DW & \xrightarrow{f'} & D'X \\
\downarrow{g'} & & \downarrow{f''} \\
D''Y & \xrightarrow{g''} & Z,
\end{array}
\]

replace each map by a strict map to obtain a commutative square in \(Ht(Ps)\) (by Theorem 1, and §5).

\[
\begin{array}{ccc}
DW & \xrightarrow{P} & DW' \\
\downarrow{G'} & & \downarrow{F''} \\
D''Y & \xrightarrow{G''} & Z
\end{array}
\]

Here \(P\) is induced by a permutation \(\Pi\) of \(DW\). Since, for all \(k\), \(SU(2k)\) is simply connected, we may "pull back" canonical homotopies to obtain a homotopy \(P \simeq DW\). This completes the proof, since cofinal inclusions are inverted in \(Ht(Ad)\).

7. **Further properties.** \^ commutes with the suspension

\[\wedge S^1: Ht(Ad) \to Ht(Ad),\]

\wedge satisfies a Kunneth formula for stable integral homology (see [6]), and \wedge is weakly universal for pairings (see [9], [6]). We conjecture the existence of an internal mapping functor adjoint to \(\wedge\) using a suitable category of permutation maps and methods of Quillen [7].

**Added in Proof.** This follow's from Brown's Theorem by Heller [Trans. Amer. Math. Soc. 147 (1970), 573–602].

**References**


DEPARTMENT OF MATHEMATICS, Hofstra University, Hempstead, New York 11550