EIGENFUNCTION EXPANSIONS FOR NONDENSELY DEFINED OPERATORS GENERATED BY SYMMETRIC ORDINARY DIFFERENTIAL EXPRESSIONS

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1. Nondensely defined symmetric ordinary differential operators. This note is a sequel to [2]; the notations are the same. Let \( L \) be the formally symmetric ordinary differential operator

\[
L = \sum_{k=0}^{n} p_k D^k = \sum_{k=0}^{n} (-1)^k D^k \bar{p}_k, \quad D = \frac{d}{dx},
\]

where the \( p_k \) are complex-valued functions of class \( C^k \) on an interval \( a < x < b \), and \( p_n(x) \neq 0 \) there. In the Hilbert space \( \mathcal{H} = L^2(a, b) \) let \( S_0 \) be the closure in \( \mathcal{H}^2 \) of the set of all \( \{f, Lf\} \) for \( f \in C_0^\infty(a, b) \), the functions in \( C^\infty(a, b) \) vanishing outside compact subintervals of \( a < x < b \). This \( S_0 \) is a closed densely defined symmetric operator whose adjoint has the domain \( \mathcal{D}(S_0^*) \), the set of all \( f \in C^{n-1}(a, b) \) such that \( f^{(n-1)} \) is absolutely continuous on each compact subinterval and \( Lf \in \mathcal{H} \). For \( f \in \mathcal{D}(S_0^*) \), \( S_0^* f = Lf \). If \( M_0 = S_0^* \ominus S_0 \), then

\[
\dim(M_0)^- = \dim \mathcal{D}((M_0)^-) = \dim \mathcal{V}(S_0^* \ominus iI) = \omega^+, \tag{1}
\]

say (\( \mathcal{V}(T) = \) null space of \( T \)). Thus \( 0 \leq \omega^- \leq n \), and \( \dim M_0 = \omega^+ + \omega^- \leq 2n \). Let \( S_0 \) be a subspace of \( \mathcal{H} \), \( \dim S_0 = p < \infty \), and define the operator \( S \), with \( \mathcal{D}(S) = \mathcal{D}(S_0) \cap (S_0 \ominus S_0) \), via \( S \subset S_0 \). We see that (2.1) of [2] is satisfied and Theorem 1 of [2] is applicable to \( S \). If \( \omega^+ = \omega^- = \omega \), which we now assume, then Theorem 2 of [2] is also applicable.

For \( u, v \in \mathcal{D}(S_0^*) \) we have Green’s formula

\[
\int_{y}^{x} \left( \bar{v}L_u - u\bar{L_v} \right) = [uv](x) - [uv](y),
\]

where \([uv]\) is a semibilinear form in \( u, u', \ldots, u^{(n-1)} \) and \( v, v', \ldots, v^{(n-1)} \). From this it follows that \([uv](x)\) tends to limits \([uv](a), [uv](b)\) as \( x \) tends to \( a, b \). Then we may write

\[
\langle uv \rangle = (Lu, v) - (u, Lv) = [uv](b) - [uv](a).
\]


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Thus, in Theorem 2 of [2], (ii) represents a set of boundary-integral conditions, and (iii) (or the expression for $H_s$) shows that both boundary and integral terms appear in the expression for the operator part of $H$.

2. Eigenfunction expansions. For any selfadjoint subspace extension $H = H_s \oplus H_\infty$ of $S$ in $\mathcal{H}_2$, as given in Theorem 2 of [2], we have $H_s = \int_{-\infty}^{\infty} \lambda \, dE_\lambda(\lambda)$, where $\{E_\lambda(\lambda)\}$ is the spectral family of projections in $\mathcal{H} \oplus H(0)$ for $H_s$. We can explicitly describe the $E_\lambda(\lambda)$ in terms of a basis for the solutions of $(L - \ell)u = \phi \in \mathcal{H}_0$, $\ell \in \mathbb{C}$. Let $\phi_1, \ldots, \phi_p$ be an orthonormal basis for $\mathcal{H}_0$, and let $c$ be fixed, $a < c < b$. Let $s_j(x, \ell)$, $a < x < b$, $\ell \in \mathbb{C}$, $j = 1, \ldots, n + p$, satisfy

$$(L - \ell)s_j = 0, \quad s^{(k-1)}(c, \ell) = \delta_{jk}, \quad j, k = 1, \ldots, n,$$

(2.1)

$$(L - \ell)s_{n+j} = \phi_j, \quad s^{(k-1)}(c, \ell) = 0, \quad j = 1, \ldots, p, \quad k = 1, \ldots, n.$$
3. **Systems of differential operators.** The results in Theorems 1 and 2 carry over to $S$ generated by a system of ordinary differential operators. We indicate the situation for a first order system. Let $L = P_1 D + P_0$, where $P_1, P_0$ are $m \times m$ matrix-valued functions on $a < x < b$, with $P_1 \in C^1(a, b), P_0 \in C(a, b)$, and $P_1^{-1}(x)$ existing for $a < x < b$. Thus $L$ operates on vector-valued functions considered as $m \times 1$ matrices. We assume $L$ is formally symmetric, i.e., $P_1^* = -P_1, P_0^* = P_0^*$. The relevant Hilbert space is $\mathcal{H} = L_m^2(a, b)$, the set of all $m \times 1$ matrix-valued functions $u$ on $a < x < b$ such that $(u, u) < \infty$. In general, for any two matrix-valued functions $F, G$ such that $G^*F$ is defined and can be integrated, we define $(F, G) = \int_a^b G^*F$. The domain of the operator $S_0$ consists of all $f \in \mathcal{H}$ which are absolutely continuous on each compact subinterval, and $Lf \in \mathcal{H}$; for $f \in \mathcal{D}(S_0^*), S_0f = Lf$. Green’s formula in this case gives

$$\int_a^b v^*Lu - (Lv)^*u = [uv](x) - [uv](y),$$

where $[uv](x) = v^*(x)P_1(x)u(x)$. The operator $S_0 \subset S_0^*$ has a domain consisting of all $f \in \mathcal{D}(S_0^*)$ such that $\langle fg \rangle = 0$ for all $g \in \mathcal{D}(S_0^*)$, where $\langle fg \rangle = (Lu, v) - (u, Lv)$. For $M_0 = S_0^* \ominus S_0$ we have $0 \leq \text{dim } M_0 \leq 2m$. If $\mathcal{H}_0 \subset \mathcal{H}$, $\mathcal{H}_0 = \mathcal{H}$, we can define $S$ as in (2.2) of [2], and then (2.1) of [2] is valid. Theorems 1 and 2 of [2] can then be applied.

We describe concretely the regular case where $a, b$ are finite, $P_1, P_0$ are continuous on the closed interval $a \leq x \leq b$, and $P_1^{-1}(x)$ exists there. Then $\mathcal{D}(S_0^*)$ is the set of all $f \in \mathcal{H}$ which are absolutely continuous on $a \leq x \leq b$ and $Lf \in \mathcal{H}$; and $\mathcal{D}(S_0)$ is the set of those $f \in \mathcal{D}(S_0^*)$ satisfying $f(a) = f(b) = 0$. In this case dim$(M_0)^\pm = m$, and Theorem 2 of [2] takes the following form.

**THEOREM 3.** In the regular case of a first order system $L$ as given above, let $H$ be a selfadjoint subspace extension of $S$ in $\mathcal{H}_2$, with dim $H(0) = s$. Let $\varphi_1, \ldots, \varphi_p$ be an orthonormal basis for $\mathcal{H}_0$, with $\varphi_1, \ldots, \varphi_s$ a basis for $H(0)$. Then $H = \{h, Lh + \varphi\}$ such that $h \in \mathcal{D}(S_0^*), \varphi \in \mathcal{H}_0$, and satisfying

(i) $\langle h, \Phi_0 \rangle = 0,$

(ii) $Mh(a) + Nh(b) + (h, Z) = 0,$

(iii) $\varphi = \Phi_0c + \Phi_1[(h, \Psi) + Ch(a) + Dh(b)],$

where $\Phi_0, \Phi_1$ are matrices with columns $\varphi_1, \ldots, \varphi_s$ and $\varphi_{s+1}, \ldots, \varphi_p$ respectively; $c, M, N, C, D$ are matrices of complex constants of order $s \times 1, m \times m, m \times m, (p - s) \times m, (p - s) \times m$ respectively, and

(a) rank$(M : N) = m,$

(b) $MP_1^{-1}(a)M^* - NP_1^{-1}(b)N^* = 0,$
(c) \( \Psi = \Phi_1 \{ E + \frac{1}{2} [DP_1^{-1}bD^* - CP_1^{-1}(a)C^*] \}, E = E^* \),
(d) \( Z = \Phi_1 [DP_1^{-1}bN^* - CP_1^{-1}(a)M^*] \).

Conversely, if there exist \( M, N, C, D, E \) satisfying (a), (b) and \( \Psi, Z \) are defined by (c), (d), then \( H \) defined by (i)–(iii) is a self-adjoint extension of \( S \) with \( \dim H(0) = s \). The operator part \( H_s \) of \( H \) is

\[
H_sh = Lh - \Phi_0 (Lh, \Phi_0) + \Phi_1 [(h, \Psi) + Ch(a) + Dh(b)].
\]

Here \( (M : N) \) is an \( m \times 2m \) matrix obtained by setting the columns of \( M \) next to those of \( N \) in the order indicated, and \( E \) is a \( (p - s) \times (p - s) \) matrix of constants. The operator extensions \( H \) are those given by the case \( s = 0 \), and these properly include those studied by A. M. Krall [3, Theorem 5.1]. He considered the operator cases when \( P_1(x) = -iI \), and \( \Psi = 0, E = 0 \), i.e., only those operators \( H \) which do not contain an integral term in the operator. (In his condition (5.5), p. 444 of [3], which is the analog of (d) above, \( -i \) should be replaced by \( +i \).)

The analogs of the expansion results, Theorems 1 and 2, are valid for the general singular case. Let \( s_j(x, \ell), a < x < b, \ell \in \mathcal{C}, \) satisfy \((L - \ell)s_j = 0, s_j(c, \ell) = e_j \) for \( j = 1, \ldots, m \), and \((L - \ell)s_{m+j} = \phi_j, s_{m+j}(c, \ell) = 0 \) for \( j = 1, \ldots, p \), where \( a < c < b \) and \( e_j \) is the unit vector with 1 in the \( j \)th row. Let \( S(x, \ell) \) be the matrix with columns \( s_1(x, \ell), \ldots, s_{m+p}(x, \ell) \).

**Theorem 4.** Let \( L \) be a first order system, and \( H = H_s \oplus H_\infty \) a self-adjoint extension of \( S \) in \( \mathcal{S}^2 \), \( \mathcal{S} = \mathcal{L}_2(a, b) \), with \( H_s = \{ \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda) \} \) in \( \mathcal{S} \oplus H(0) \). There exists an \( (m + p) \times (m + p) \) matrix-valued function \( \rho \) on \( \mathbb{R} \), which is Hermitian, nondecreasing, and of bounded variation on each finite interval. If \( \Delta = (\mu, \lambda) \), and \( \mu, \lambda \) are continuity points of \( E_s \), then for \( f \in C_0(a, b) \cap (\mathcal{S} \oplus H(0)) \),

\[
E_s(\Delta) f(x) = \int_{\Delta} S(x, v) \, d\rho(v) \hat{f}(v), \quad \hat{f}(v) = (f, S(v)).
\]

If \( f \in \mathcal{S} \oplus H(0) \), then \( \hat{f} \in \mathcal{L}^2(\rho) \), \( \| f \| = \| \hat{f} \| \), and

\[
f(x) = \int_{-\infty}^{\infty} S(x, v) \, d\rho(v) \hat{f}(v).
\]

4. **Selfadjoint extensions in larger spaces.** In either the \( n \)th order case or first order system case, if \( \dim (M_0)^+ \neq \dim (M_0)^- \) there are no selfadjoint extensions of \( S \) in \( \mathcal{S}^2 \). However, there always exist such extensions in a larger space \( (\mathcal{S} \oplus \mathcal{R})^2 \), where \( \mathcal{R} \) is a Hilbert space. Let \( H = H_s \oplus H_\infty \) be any such with \( H_s = \{ \int_{-\infty}^{\infty} \lambda \, dE_s(\lambda) \} \) on \( (\mathcal{S} \oplus \mathcal{R}) \oplus H(0) \). Let \( P \) be the orthogonal projection of \( \mathcal{S} \oplus \mathcal{R} \) onto \( \mathcal{S} \), and define \( F_s(\lambda)f = PE_s(\lambda)f \), for \( f \in \mathcal{S} \oplus PH(0), \lambda \in \mathbb{R} \). The proofs of Theorems 1, 2, 4 involve a
nontrivial adaptation of the method used in our earlier paper on operators
[1], and we can avoid the use of the results of A. V. Štraus mentioned
there. Thus we can show that these theorems are valid for any $H$ in
$(\mathcal{H} \oplus \mathcal{K})^2$, with $E_s$ replaced by $F_s$, and $\mathcal{H} \oplus H(0)$ replaced by $\mathcal{H} \oplus PH(0)$. Hence it is not necessary to assume $\dim(M_0)^+ = \dim(M_0)^-$.

Detailed proofs will appear elsewhere.

REFERENCES


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