TOPOLOGICAL INVARIANCE OF CERTAIN COMBINATORIAL CHARACTERISTIC CLASSES

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(X, \partial X) denotes a finite polyhedral pair which is a rational homology manifold pair. \sigma denotes an additive invariant associated to nonsingular quadratic forms over the rationals, e.g., the index the discriminant. In this note we prove what the title says, for certain combinatorial invariants \gamma(X, \sigma) associated to X.

The classes \gamma(X, \sigma), which generalize the combinatorial Pontrjagin classes, occur as one of the two following types:

(a) If the additive invariant \sigma is the index, then

\gamma(X, \sigma) \in K_x^{G/TOP}(X, \partial X)^{1}

where x = \dim(X); for X a PL manifold, \gamma(X, \sigma) localized away from 2 coincides with the KO_* orientation class for X defined in [8]; and \gamma(X, \sigma) \otimes \mathbb{Q} is equivalent to the PL Pontrjagin classes.

(b) If \sigma has finite exponent, then \gamma(X, \sigma) \in \sum_i H_{4i+2}(X, \partial X), \mathbb{Z}_4). Rational manifolds are the only possible fixed point sets of PL actions of groups of prime order on manifolds [6]. If X is the fixed point set of such an action then the bocksteins of certain of the exponent four classes \gamma(X, \sigma) must vanish [3].

For any closed, rational homology manifold Y, \sigma(Y) will denote the evaluation of \sigma on the mid-dimensional intersection pairing of H_* (Y, \mathbb{Q}). Note that \sigma(Y) = 0 if dim(Y) \neq 0 (4). Let \{P\} denote the set of subpolyhedra in X \times D_L (L = large) which have either linear normal bundles or linear normal bundles with "Z^-type" singularities [4], [7]. It is an important theorem that the classes \gamma(X, \sigma) can be identified with the geometric construction \{P\} \rightarrow \{\sigma(P)\} (see [7], [8], and compare with [3]).

Let (X, \partial X), (X', \partial X') denote finite polyhedral pairs which are rational homology manifold pairs.

**Theorem.** If f:(X, \partial X) \rightarrow (X', \partial X') is a topological homeomorphism then

\[ f_* (\gamma(X, \sigma)) = \gamma(X', \sigma). \]


1 G/TOP is the torsion free (in homotopy) H-space factor of G/TOP, with respect to the "characteristic variety" H-space structure for G/TOP [7]. K_*^{G/TOP}( ) denotes the homology theory having G/TOP as its zeroth loop spectrum.
PROOF. It will suffice to consider \( \sigma(P) \) for those polyhedra \( P \subset X \times D^L \) which have \( P \times D^m \) for a regular neighborhood.

Following Novikov [4], let \( T^{m-1} \times I \subset D^m - \partial D^m \) denote the standard embedding of the \((m - 1)\)-torus, crossed with the unit interval, into the open \( m \)-ball. Consider the restriction

\[
f(P \times T^{m-1} \times I^0) \to f(P \times T^{m-1} \times I^0).
\]

\( f(P \times T^{m-1} \times I^0) \) has an "end" \( E \) in the finite CW category, because \( P \times T^{m-1} \times I^0 \) does and \( f(P \times T^{m-1} \times I^0) \) is properly homotopically equivalent to \( P \times T^{m-1} \times I^0 \) under \( f \) (see [5], [9]).

By adding the end \( E \) to \( f(P \times T^{m-1} \times I^0) \), outside a compact rational homology manifold neighborhood \( R \) for \( f(P \times T^{m-1} \times 1/2) \) in \( f(P \times T^{m-1} \times I^0) \), a CW complex triple \((Y, \partial_+ Y, \partial_- Y)\) is constructed satisfying

(i) \((Y, \partial_+ Y, \partial_- Y)\) is homotopy equivalent to \( P \times T^{m-1} \times (I, 0, 1) \).

(ii) \( R^0 \) is contained in \( Y \) as an open set, and the orientation class for \((Y, \partial Y)\) restricts on \((R, \partial R)\) to the orientation class for \((R, \partial R)\).

Finally by putting the composition map

\[
(Y, \partial \pm Y) \sim P \times T^{m-1} \times (I, \partial \pm I) \xrightarrow{P_2 \times P_3} T^{m-1} \times (I, \partial \pm I)
\]

in transverse position to \( T^{m-1} \times 1/2 \) (see (ii) above), we obtain a "cobordism" \( W \) from \( P \times T^{m-1} \) to a polyhedron \( L \) which is a PL collared subset of \( f(P \times T^{m-1} \times I^0) \). Note that there is a canonical map \( h: W \to T^{m-1} \), and that \( \gamma(X, \sigma) \) is computed "on \( P \)" as \( \sigma(h_{\partial_1}^{-1} w(t_0)) \), where \( t_0 \in T^{m-1} \). The corresponding computation for \( \gamma(X', \sigma) \) is \( \sigma(h_{\partial_1}^{-1} w(t_0)) \).

To complete the proof of the theorem it must be shown that \( \sigma(h_{\partial_1}^{-1} w(t_0)) = \sigma(h_{\partial_1}^{-1} w(t_0)) \). We do this by constructing a rational Poincaré duality cobordism from \( h_{\partial_1}^{-1} w(t_0) \) to \( h_{\partial_1}^{-1} w(t_0) \). First note that \( W \) is actually a Poincaré cobordism with respect to the coefficients \( Q(\pi_1(T^{m-1})) \) (see (i), (ii) above). Use the PL rational homology manifolds structures of \( \partial_{\pm} W \) to put \( h_{\partial_{\pm} W} \) in transverse position, simplex by simplex to the sequence \( t_0 \subset T^1 \subset T^2 \subset T^3 \subset \cdots \subset T^{m-2} \subset T^{m-1} \). There is one surgery obstruction, \( S(h, \partial h) \), to extending this sequential transversality to all of \( h \) in the category of codimension one nested spaces which are Poincaré with respect to the nested coefficients \( Q \subset Q(\pi_1(T^1)) \subset \cdots \subset Q(\pi_1(T^{m-1})) \) (see §7.11 of [2]).

It only remains to see \( S(h, \partial h) = 0 \). It is helpful to consider \( S(h, \partial h) \) in the following simple (but, by the constructions of [2], universally typical) case. \( M, N \) are two, compact, differentiable manifolds with dimensions.
Let the maps \( \partial_0 M \subset M \xrightarrow{h_M} T^{m-1} \leftarrow h_N N \supset \partial_0 N \) induce isomorphisms of fundamental groups. \( g: \partial_0 M \to \partial_0 N \) is a homology equivalence with respect to the coefficients \( Q(\pi_1(T^{m-1})) \), and \( g \) commutes with \( h_M, h_N \). Let \( h: W \to T^{m-1} \) equal the union along \( g \) of \( h_M \) and \( h_N \). A transversality of \( h \mid W \) to \( t_0 \subset T^1 \subset T^2 \subset \cdots \subset T^{m-1} \) extends to all of \( h \) if \( g: \partial_0 M \to \partial_0 N \) can be made transversal to

\[
h^{-1}_g(t_0 \subset T^1 \subset \cdots \subset T^{m-1})
\]

in such a way that

\[
g: g^{-1}(h \mid W)(t_0 \subset T^1 \subset \cdots \subset T^{m-1}) \to h^{-1}_g(t_0 \subset T^1 \subset \cdots \subset T^{m-1})
\]

is a homology equivalence with respect to the nested coefficients \( Q \subset Q(\pi_1(T^1)) \subset \cdots \subset Q(\pi_1(T^{m-1})) \). This is precisely what the “rational form” of the Farrel-Hsiang splitting theorem allows [1]. It might be necessary to first vary \( g: \partial_0 M \to \partial_0 N \) through a cobordism which is a homological \( H \)-cobordism with respect to the coefficients \( Q(\pi_1(T^{m-1})) \) before achieving the desired transversality of \( g \). But such a variation is allowed in the argument of the previous paragraph. Q.E.D.

BIBLIOGRAPHY