THE HOMEOMORPHISM PROBLEM FOR $S^3$

BY JOAN S. BIRMAN$^1$ AND HUGH M. HILDEN$^2$

Communicated by William Browder, January 22, 1973

1. Introduction. Let $M$ be a closed, orientable 3-manifold which is defined by a Heegaard splitting of genus $g$. Each such Heegaard splitting may be associated with a self-homeomorphism of a closed, orientable surface of genus $g$ (the surface homeomorphism is used to define a pasting map) and it will be assumed that this surface homeomorphism is given as a product of standard twist maps [3] on the surface. We assert:

THEOREM 1. If $M$ is defined by a Heegaard splitting of genus $\leq 2$, then an effective algorithm exists to decide whether $M$ is topologically equivalent to the 3-sphere $S^3$. This algorithm also applies to a proper subset of all Heegaard splittings of genus $> 2$.

This result is of interest because it had not been known whether such an algorithm was possible for $g \geq 2$, and also because the algorithm has a possible application in testing candidates for a counterexample to the Poincaré conjecture.

In this note we will describe the algorithm, and sketch a brief proof. Related results about the connections between representations of 3-manifolds as Heegaard splittings, and as branched coverings of $S^3$, are summarized at the end of this paper. A detailed report will appear in another journal.

2. The algorithm. Let $X_g$ be a handlebody of genus $g \geq 0$ which is imbedded in Euclidean 3-space as illustrated in Figure 1. Let $X'_g$ be a

![Figure 1. The Handlebody $X_g$](image)


$^1$ The work of the first author has been supported in part by NSF grant #GP-34324X.

$^2$ The work of the second author has been supported in part by NSF grant #GP-34059.
second handlebody, which is so related to $X_g$ that a translation $\tau$ parallel
to the x-axis maps $X_g$ onto $X'_g$. Let $\Phi$ be a homeomorphism of $\partial X_g \to \partial X'_g$.
Let $M = X_g \cup \Phi X'_g$ be the 3-manifold which is obtained by identifying
the boundaries of $X_g$ and $X'_g$ according to the rule $\tau \Phi(z) = z$, $z \in \partial X_g$.
Every closed 3-manifold $M$ admits such a representation.

Let $c$ be a simple closed curve on $\partial X_g$, and let $\gamma_c$ be a twist about $c$ (see
[3], [4]). It was proved in [4] that if $g > 0$, then every homeomorphism
of $\partial X_g \to \partial X_g$ is isotopic to a product of twists $\gamma_{c_i}$ about the curves $c_i,
1 \leq i \leq 3g - 1$, in Figure 1.\(^3\) We will make the assumption that our
homeomorphism $\Phi$ is given as a product of the particular twists $\gamma_{c_1}, \ldots ,
\gamma_{c_{2g+1}}$. This involves no loss in generality if $g \leq 2$, but if $g > 2$ the class
of maps $\Phi$ which can be so represented is somewhat restricted. We are now
ready to state the algorithm for deciding whether $M = X_g \cup \Phi X'_g$ is
homeomorphic to $S^3$.

**Step 1.** Given the homeomorphism

$$\Phi = \gamma_{c_{\mu_1}}^{\varepsilon_1} \cdots \gamma_{c_{\mu_r}}^{\varepsilon_r}$$

where each $\varepsilon_i = \pm 1$, and each $\mu_i$ is between 1 and $2g + 1$, construct a
diagram of the $(2g + 2)$-string braid

$$\beta = \sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$$

where $\sigma_i$ is a standard generator of the braid group (see [1]). The braid $\sigma_i$
is illustrated in Figure 2.

![Figure 2. The Braid $\sigma_i$](image)

**Step 2.** Using the braid $\beta$, construct a link $L$, given in projection, by
joining the ends of the braid $\beta$ in pairs according to the rule illustrated in
Figure 3. The top of string $2i + 1$ is connected to the top of string $2i + 2$, for
$i = 0, \ldots, g$; the bottom of string $2i + 1$ is connected to the bottom
of string $2i + 2$ for each $i = 0, \ldots, g$. The resulting link is said to be
displayed as a "plat".

\(^3\) If $g = 0$ every homeomorphism $\Phi$ is isotopic to the identity map, the set \{c\} is empty,
and $M \sim S^3$. If $g = 1$, the twist maps $\gamma_{c_1}$ and $\gamma_{c_2}$ are isotopic, hence only two twist maps
$\gamma_{c_1}$ and $\gamma_{c_2}$ are needed.
Step 3. Verify (by checking the projection) whether the plat $L$ has multiplicity 1. This is a necessary condition for $M \sim S^3$. If so, apply the algorithm given by Haken in [2], or by Schubert in [5], to decide whether $L$ is the trivial knot. We assert that $M \sim S^3$ if and only if $L$ is the trivial knot.

3. Sketch of proof. We can assume without loss in generality that the embedding of $X_g$ and $X'_g$ in 3-space $\mathbb{E}^3$ is chosen in such a way that both $X_g$ and $X'_g$ are invariant under a rotation $\Omega$ of $180^\circ$ about the $x$-axis. There is also no loss in generality in assuming that the twist maps $\gamma_{c_1}, \ldots, \gamma_{c_{2g+1}}$ are defined in such a way that each $\gamma_{c_i}$ commutes with the rotation $\Omega$. Since the translation $\tau$ likewise commutes with $\Omega$, it follows that

$$(\tau \Omega) = \Omega (\tau \Omega).$$

Let $M/\Omega$ be the orbit space of $M = X_g \cup_\partial X'_g$ under the action of $\Omega$, and let $\rho$ be the natural projection from $M$ to $M/\Omega$. The condition (3) ensures that $M/\Omega$ is well defined. The quotient spaces $X_g/\Omega$ and $X'_g/\Omega$ are each homeomorphic to 3-balls, hence

$$(4) \quad M/\Omega = (X_g/\Omega) \bigcup_{\rho \Phi \rho^{-1}} (X'_g/\Omega)$$

is represented as a genus zero Heegaard splitting, hence $M/\Omega$ must be homeomorphic to $S^3$. Thus the triplet $(\rho, M, M/\Omega)$ exhibits $M$ as a 2-sheeted branched covering of $S^3$. The branching set is the image under $\rho$ of the fixed point set of $\Omega$, that is of the set $(X_g \cap x$-axis) $\cup$ $(X'_g \cap x$-axis).

To understand the structure of the branching set, observe that the surface homeomorphism $\rho \Phi \rho^{-1}$ which defines the Heegaard splitting of $M/\Omega$ is a homeomorphism of $S^2 \to S^2$, and hence it is isotopic to the identity. This isotopy can be used to define a homeomorphism $F$ of $M/\Omega \to M/\Omega$, and it can be shown that the image of the fixed point set of $\Omega$ under the product $F \rho$ is precisely the link $L$ described in Steps 1 and 2 of the algorithm.

**FIGURE 3.** $(2g+2)$-STRING PLAT
Suppose that $M$ is homeomorphic to $S^3$. Then by a theorem of Waldhausen [7] the fixed point set of $\Omega$ must be the trivial knot, hence its image under $F\rho$ must also be trivial. Therefore a necessary condition for $M \sim S^3$ is that $L$ have a single, unknotted component. The algorithm given in [2] and [5] enables us to test whether $L$ is, in fact, trivial. If it is trivial, then $M$ is the 2-fold branched covering of $S^3$ branched over the trivial knot. But then, $M \sim S^3$, hence the condition is also sufficient.

We remark that if Waldhausen's result [7] could be extended to transformations of period $p > 2$, then our algorithm could be extended to the class of all 3-manifolds which admit representations as $p$-fold branched cyclic coverings of $S^3$. It is not known whether this includes all closed 3-manifolds.4

4. Related results. The Heegaard genus of a 3-manifold $M$ is the smallest integer $g$ such that $M$ admits a Heegaard decomposition $X_g \cup_\phi X'_g$. The bridge number $b$ of a link $L$ is the smallest integer $n$ such that $L$ can be exhibited in a $b$-bridge presentation [6]. The braid number $n$ of a link $L$ is the smallest integer $n$ such that $L$ can be represented as a closed braid with $n$-strings [1]. (This is not the same as a "plat".)

**Corollary 1.** Every 3-manifold of Heegaard genus $g \leq 2$ can be exhibited as a 2-fold branched cyclic covering of $S^3$, branched over a knot or link of bridge number $g + 1$. The two-fold branched cyclic covering of $S^3$ branched over a knot or link of bridge number $b$ is a 3-manifold of Heegaard genus $\leq b - 1$. (This generalizes a result due to Schubert [6].)

**Theorem 2.** The $p$-fold branched cyclic covering of $S^3$, branched over a knot of braid number $n$, is a 3-manifold of Heegaard genus $\leq (p - 1)(n - 1)$, for every $p \geq 2$.

**References**


4 A new result of J. Montisinos establishes that this does not include all closed 3-manifolds. See J. Montisinos, *3-Variétés qui ne sont pas revêtements cycliques ramifiés sur $S^3$*, (to appear).

DEPARTMENT OF MATHEMATICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY 07030

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HAWAII 96822

Current address (Joan Birman): Department of Mathematics, Columbia University, New York, New York 10027.