Let $X$ be a connected $C^\infty$ manifold. Denote by $P(X)$ the total space of piecewise smooth paths in $X$. Choose a base point $x_0$. Denote by $P(X; x_0)$ (resp. $\Omega X$) the space of piecewise smooth paths (resp. loops) from the base point $x_0$.

Let $k$ be the field of real (or complex) numbers. All differential forms are $k$-valued. Let $\omega_1, \omega_2, \ldots$ denote 1-forms on $X$. For a piecewise smooth path $\alpha: I \to X$, let $f_1(t) = \omega_i(\alpha(t), \dot{\alpha}(t))$ be the value of the 1-form $\omega_i$ at the tangent vector $\dot{\alpha}(t)$ of $X$. Define the $r$-time iterated integral $\int w_1 \cdots w_r$ to be the $k$-valued function on $P(X)$ whose value at $\alpha$ is given by

$$\left\langle \int w_1 \cdots w_r, \alpha \right\rangle = \int_0^1 \int_0^{t_2} \cdots \int_0^{t_r} f_1(t_1) dt_1 \cdots f_{r-1}(t_{r-1}) dt_{r-1} f_r(t_r) dt_r$$

when $r > 0$ and $= 1$ when $r = 0$. At times, we shall also take $\int w_1 \cdots w_r$ as its restriction on $\Omega X$ or $P(X; x_0)$.

Let $F$ be the function algebra on $P(X)$ consisting of those functions whose value at each path $\alpha$ remains invariant under any piecewise smooth homotopy of $\alpha$ relative to $I$. In this note, we shall consider the subspace of $F$ whose elements are linear combinations of iterated integrals. A characterization of this subspace in terms of the fundamental group $\pi_1(X)$ will be given.

We begin with a differential graded subalgebra $A$ of the exterior algebra $\Lambda(X)$. The following assumptions are made:

I. $dA^0 = A^1 \cap dA^0(X)$.

II. $\dim H^1(A) < \infty$.

III. The canonical homomorphism $H^q(A) \to H^q(X; k)$ is an isomorphism when $q = 1$ and is a monomorphism when $q = 2$.

A primary example is the case of $A = \Lambda(X)$.

For $s \geq 0$, denote by $F_s$ the subspace of $F$ whose elements are linear combinations of iterated integrals of the type...
Then $k = F_d(0) \subset \cdots \subset F_d(s) \subset \cdots$. Moreover $F_A = \bigcup F_d(s)$ turns out to be closed under multiplication. Let $F_A'(s)$ (resp. $F_A''(s)$) be obtained from $F_d(s)$ by restricting to $\Omega X$ (resp. $P(X; x_0)$). Then $F_A' = \bigcup F_d(s)$ and $F_A'' = \bigcup F_d(s)$ are algebras obtained from the algebra $F_A$ by restrictions.

Let $k\pi_1(X)$ be the group algebra of $\pi_1(X)$ over $k$, and let $N$ be the augmentation ideal, which is generated by all $\langle \alpha \rangle - 1$, where $\langle \alpha \rangle$ denotes the homotopy class of a piecewise smooth loop at $x_0$. There is a pairing $F_A' \times k\pi_1(X) \to k$ given by $(u, \langle \alpha \rangle) \mapsto \langle u, \alpha \rangle$ which is the value of the linear combination $u$ of iterated integrals at the loop $\alpha$.

**THEOREM 1.** With respect to the above pairing $F_A'(s) = (N^{s+1})^\perp$, $s \geq 0$.

In order to outline a proof of this theorem, we recall that iterated integrals can be defined for forms of higher degrees in $A$. In relation to $\Omega X$, such iterated integrals form a differential graded algebra $A'$ with an ascending filtration $\{A'(s)\}$ (see [3]). Choose a suitable cubical chain complex $C_*(\Omega X)$ so that it has a descending filtration by the powers of its augmentation ideal. Let $\{B(s)\}$ be the dual ascending filtration for the cochain complex $C^*(\Omega X; k)$. Then $B = \bigcup B(s)$ is a filtered subcomplex of $C^*(\Omega X; k)$. Theorem 1 follows from comparing the spectral sequences of the filtered cochain complexes $A'$ and $B$ and the fact that $F_A' \approx H^0(A')$. Since the restriction map $F_A' \to F_A''$ is surjective, we are also led to the next conclusion.

**THEOREM 2.** If $A^0$ separates points of $X$ and if $\bigcap N^s = 0$, then $F_A''$, taken as an algebra of functions on the universal covering space $\widehat{X}$ of $X$, separates points of $\widehat{X}$.

This result is related to a work of Parsin [5]. He considered the case where $X$ is a Riemann surface, and $A$ is the algebra of holomorphic differential forms. Our assumption III does not hold for his case.

If $\pi_1(X)$ is finitely generated torsion free nilpotent, we can show that $F_A'$ is the coordinate ring of a simply connected nilpotent Lie group $G$ having a uniform discrete subgroup $\Gamma \approx \pi_1(X)$. By constructing an injection $F_A' \to F_A''$, we obtain the next assertion.

**THEOREM 3.** If $X$ is a connected $C^\infty$ manifold with $\pi_1(X)$ being finitely generated torsion free nilpotent, then there exists a compact nilmanifold $M(X)$ and a $C^\infty$ map $X \to M(X)$ which induces an isomorphism for the fundamental groups.

Observe that $G$ can be taken as the Malcev completion of $\Gamma \approx \pi_1(X)$. 
The nilmanifold $M(X) = G/T$ associated to $X$ is essentially unique (see [1] or [6]).

**Remark.** In the case of $X$ being Riemannian, there is a canonical injection $F'_A \to F''_A$ with $A = \Lambda(X)$ so that, in the theorem, the map $X \to M(X)$ is canonical. If $X$ is, moreover, real analytic, so is the map.

The details of these theorems in a somewhat more general context will appear elsewhere.

**BIBLIOGRAPHY**


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