I. Introduction. Let $G$ denote a locally compact abelian (LCA) group and let $\Gamma$ denote the group which is dual to $G$. If $1 \leq p \leq \infty$, let $L_p(G)$ (or $L_p(\Gamma)$) denote the space of $p$-power integrable functions with respect to Haar measure on $G$ (or on $\Gamma$); let $C(G)$ denote the algebra of bounded continuous functions on $G$ and let $C_0(G)$ consist of those functions in $C(G)$ which vanish at infinity. In [2] and [3] Figa-Talamanca and Figa-Talamanca and Gaudry studied the $p$-Fourier algebra

$$A_p(\Gamma) = [L_p(\Gamma) \hat{\otimes} L_p(\Gamma)]/K$$

where $1 \leq p < \infty$, $p'$ is conjugate to $p$, $\hat{\otimes}$ is the projective tensor product, and $K$ denotes the kernel of the convolution operator $c : L_p \otimes L_{p'} \to C_0(\Gamma)$ by $c(f \otimes g)(y) = f \ast g(y)$. $A_p(\Gamma)$ carries the quotient norm. Herz [7] showed that $A_p(\Gamma)$ is a Banach algebra under pointwise multiplication. In [2] Figa-Talamanca showed that the dual space of $A_p(\Gamma)$ is isometrically isomorphic to the space $M_p(\Gamma)$ of bounded, translation invariant, linear operators on $L_p(\Gamma)$ and that the weak operator topology on $M_p(\Gamma)$ and the weak*-topology agree on bounded sets. Implicit in [2] is the fact that $A_2(\Gamma)$ is isometrically isomorphic with $A(\Gamma)$, the algebra of Fourier transforms of integrable functions on $G$; $A(\Gamma)$ is equipped with the inherited norm; see also [1]. Hewitt’s factorization theorem is used in [3] to prove that $A_1(\Gamma)$ is $C_0(\Gamma)$ ($C(\Gamma)$ when $\Gamma$ is compact).

Let $B_p(\Gamma)$ denote the algebra of functions $f$ in $C(\Gamma)$ which satisfy: $f(\gamma)h(\gamma) \in A_p(\Gamma)$ whenever $h \in A_p(\Gamma)$. $B_p(\Gamma)$ is a commutative and semi-simple Banach algebra under pointwise addition and multiplication when it is equipped with the operator norm; $B_p(\Gamma)$ is the algebra of bounded multipliers on $A_p(\Gamma)$. If $1 < p < \infty$ and if $p'$ denotes the index conjugate to $p$, then $A_{p'} = A_p$ and $B_{p'} = B_p$, so that we may restrict $p$ to $1 < p < 2$. It is easy to see that $B_1(\Gamma) = C(\Gamma)$, and Helson’s theorem [11, p. 73] says that $B_2(\Gamma)$ is the algebra of Fourier transforms of bounded measures on $G$ with the inherited norm. Since the inclusions $A_2(\Gamma) \subset A_p(\Gamma) \subset A_1(\Gamma)$ are continuous if $1 < p < 2$, it follows that the maximal


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ideal space of $A_p(\Gamma)$ can be identified with $\Gamma$. Thus $B_p(\Gamma)$ is an algebra of multipliers in the sense of Larsen [9]; we refer to this monograph for the basic facts regarding multiplier algebras.

The purpose of this note is to show that the only complex valued functions of a complex variable which operate on (the Gel'fand transforms of) $B_p(\Gamma)$ are entire functions when $\Gamma$ is not compact. If $\Gamma$ is compact, $A_p(\Gamma) = B_p(\Gamma)$ and the class of functions which operates on $B_p(\Gamma)$ is less restrictive. When $\Gamma$ is not compact, this will imply that the algebra $B_p(\Gamma)$ is not selfadjoint and not regular and that $\Gamma$ is not dense in the maximal ideal space of $B_p(\Gamma)$. The basic result (Theorem 1, below) from which this information follows can also be used to describe the isometric isomorphisms of $A_p(\Gamma)$ onto $A_p(\Lambda)$ when $\Lambda$ is a second LCA group. A detailed development of these topics will be given in [5].

II. Functions operating on multipliers. In [6] Hahn proved that if $1 \leq p \leq 2$ and if $f \in L_p(\Gamma)$ and $g \in L_p(\Gamma)$, then $h(y) = f \ast g(y)$ is the Fourier transform of an operator $T_h$ in $M_r(G)$ if $|1/r - 1/2| \leq |1/p - 1|$ for which $\|T_h\|_r \leq \|f\|_p \|g\|_p$. We have the following extension of this result.

**Theorem 1.** Let $1 \leq p \leq 2$ and let $|1/r - 1/2| \leq |1/p - 1|$. Then $f$ in $B_p(\Gamma)$ is the Fourier transform of an operator $T_f$ in $M_r(G)$ and the map $f \rightarrow T_f$ faithful, norm decreasing representation of $B_p(\Gamma)$ as an algebra of bounded, translation invariant operators on $L_r(G)$.

It is clear that Hahn’s map $h \rightarrow T_h$ extends to a norm decreasing isomorphism of $A_p(\Gamma)$ into $M_r(G)$. Now Hahn’s Lemma 1 and the fact that $A_p(\Gamma)$ has a bounded approximate identity (since $A_2(\Gamma)$ does) can be used to extend $h \rightarrow T_h$ to all of $B_p(\Gamma)$.

The Fourier transforms of bounded measures on $G$ are elements of $B_2(\Gamma)$; denote this subalgebra by $B_2(\Gamma)$.

**Theorem 2.** Suppose that $\Gamma$ is noncompact and that $1 < p < 2$. Let $F$ be a complex valued function defined on $[-1, 1]$ for which $F(\mu(\gamma)) \in B_p(\Gamma)$ for every $\mu \in B_2(\Gamma)$ with range in $[-1, 1]$. Then $F$ admits an extension to all of $C$ as an entire function.

This follows from Igari’s Theorem 1 of [8] and from Theorem 1 above. When $\Gamma$ is compact but not discrete, $F$ must be analytic in a neighborhood of $[-1, 1]$.

Now, by reasoning as in [11, Chapter 6] or as in [8], one may conclude that

**Theorem 3.** Suppose that $\Gamma$ is noncompact and that $1 < p < 2$. Then:

1. If $F$ is a complex valued function of a complex variable for which $F(\hat{f})$ is the Gel'fand transform of a function in $B_p(\Gamma)$ whenever $\hat{f}$ is, then $F$ is an entire function.
2. For any complex number \( z \) there is a real valued function \( f \) in \( B_p(\Gamma) \) and a complex homomorphism \( h \) of \( B_p(\Gamma) \) for which \( h(f) = z \).

3. \( B_p(\Gamma) \) is not selfadjoint and not regular.

4. \( \Gamma \) is not dense in the maximal ideal space of \( B_p(\Gamma) \).

5. There is a function \( f \) in \( B_2(\Gamma) \subset B_p(\Gamma) \) with \( f \geq 1 \) on \( \Gamma \) for which \( f^{-1} \) is not in \( B_p(\Gamma) \).

The analogy between \( B_p(\Gamma) \) and \( B_2(\Gamma) \) does not end here, but we shall wait to describe the situation more completely in [5].

III. Isomorphism. Let \( \Gamma \) and \( \Lambda \) be LCA groups and let \( A_p(\Gamma) \) and \( A_p(\Lambda) \) denote their respective \( p \)-Fourier algebras for \( 1 < p < 2 \). It follows from Theorem 1 of §II and from a theorem of Strichartz [12] that the only isometric multipliers on \( A_p(\Gamma) \) are complex unit multiples of characters on \( \Gamma \). This is the basic fact needed to prove

**Theorem 4.** If \( \Phi \) is an isometric isomorphism of \( A_p(\Gamma) \) onto \( A_p(\Lambda) \), then there is a topological isomorphism \( \alpha \) of \( \Lambda \) onto \( \Gamma \) and an element \( \gamma_0 \) of \( \Gamma \) such that \( \Phi(h)(\lambda) = h(\gamma_0 \alpha(\lambda)) \).

**Corollary 4.1.** \( \Gamma \) is topologically isomorphic to \( \Lambda \) if and only if \( A_p(\Gamma) \) is isometrically isomorphic to \( A_p(\Lambda) \); i.e. \( A_p(\Gamma) \) determines \( \Gamma \).

The proof relies on the facts that \( \Phi \) extends to \( B_p(\Gamma) \), that \( \Phi \) maps isometric multipliers to isometric multipliers, and that \( G \) is topologically isomorphic to the multipliers \( \{g_0(\gamma) \mid g_0 \in G\} \) when this group is equipped with the strong operator topology.

\( A_p(\Gamma) \), \( B_p(\Gamma) \), and \( M_p(\Gamma) \) forms an interrelated system of algebras which we first studied in [4]. There, we proved Theorem 1 of §II in the context of a continuity theorem of Lévy type for \( M_p(G) \). No applications of Theorem 1 were given in [4].

**References**


