AN ILL POSED PROBLEM FOR A HYPERBOLIC EQUATION NEAR A CORNER

BY STANLEY Osher

Communicated by Eugene Isaacson, January 23, 1973

The purpose of this note is to give a simple example of an ill posed problem for a hyperbolic equation to be solved in a region whose boundary has a corner. In [2] we gave necessary and sufficient conditions for existence, uniqueness, and the validity of certain energy estimates for the solutions of a general class of these problems. Analogous conditions for problems in regions with smooth boundaries were obtained by Kreiss [1]. Our example below is somewhat unusual in that bounded $C^\infty$ initial data lead to a solution which is exponentially unbounded at the corner for any positive time.

Consider the equation

$$
\begin{pmatrix}
u \\
u_y 
\end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\
0 & 1 
\end{pmatrix} \begin{pmatrix} u \\
u_x 
\end{pmatrix} + \begin{pmatrix} 1 & 0 \\
0 & -1 
\end{pmatrix} \begin{pmatrix} u \\
u_y 
\end{pmatrix}
$$

to be solved for the complex valued functions $u$ and $v$ in the region $0 < x, y, t$ with initial conditions

(2) $u(x, y, 0) = \Phi(x, y), \quad v(x, y, 0) = \psi(x, y),$ and boundary conditions

(3) (a) $u(0, y, t) = au(0, y, t), \quad (b) v(x, 0, t) = bv(x, 0, t).$

$a$ and $b$ are complex numbers.

We have the following:

**THEOREM.** The above problem is well posed, i.e. generates a strongly continuous semigroup for $t > 0$ on $L^2$, if and only if $|ab| \leq 1$.

We note here that by the results in [1], the half space problem (1), (2) to be solved for $0 < x, t; -\infty < y < \infty$ is well posed for any boundary condition (3)(a), as is the half space problem for $0 < y, t; -\infty < x < \infty$ for any boundary condition (3)(b).

AMS (MOS) subject classifications (1970). Primary 35L50, 35L30; Secondary 78A45.

Key words and phrases. Hyperbolic equations, initial boundary conditions, well posedness, energy estimates.

1 Research supported under N.S.F. Grant No. GP29-273.
2 Fellow of the Alfred P. Sloan Foundation.
PROOF. If $|ab| > 1$, consider the functions

$$u(x, y, t) = a(ab)^{(t-x)/[2(x+y)]}, \quad v(x, y, t) = (ab)^{(t+x)/[2(x+y)]},$$

where the same argument for $ab$ is chosen in both expressions. This pair of functions satisfies the conditions of (1) and (3) with initial data which are bounded and $C^\infty$ for $x, y > 0$. The initial data are then multiplied by the factor $(ab)^{t/[2(x+y)]}$ which is exponentially unbounded, as is the solution, when $x + y \to 0$ for any positive $t$. The solution is well behaved for finite $t$ away from $x = y = 0$.

If $|ab| \leq 1$, we choose two positive numbers $c_1, c_2$ such that $c_1 |a|^2 \leq c_2$ and $c_2 |b|^2 \leq c_1$. We then have, using (1) and (3):

$$\frac{d}{dt} \int_0^\infty \int_0^\infty \left[ c_1 |u|^2 + c_2 |v|^2 \right] \, dx \, dy \leq 0.$$

$L_2$ well posedness is immediate.

REFERENCES